

UNIVERSAL R -MATRIX AND FUNCTIONAL RELATIONS

HERMAN BOOS, FRANK GÖHMANN, ANDREAS KLÜMPER,
KHAZRET S. NIROV, AND ALEXANDER V. RAZUMOV

ABSTRACT. We collect and systematize general definitions and facts on the application of quantum groups to the construction of functional relations in the theory of integrable systems. As an example, we reconsider the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ related to the six-vertex model. We prove the full set of the functional relations in the form independent of the representation of the quantum group in the quantum space and specialize them to the case of the six-vertex model.

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1. INTRODUCTION

The famous Onsager's solution [56] of the square lattice Ising model was the first essential result in the field of two-dimensional quantum integrable statistical lattice models. The next step was made by Lieb [52–55] who used the Bethe ansatz [18] to solve different partial cases of the six-vertex model. His results were generalized to the general case of the six-vertex model by Sutherland [60]. Later, Baxter proposed the method of functional relations [3–8, 10] to solve statistical models which cannot be treated with the help of the Bethe ansatz. The method works for the cases where the Bethe ansatz can be applied as well. It appears that its main ingredients, transfer matrices and Q -operators, are essential not only for the integration of the corresponding quantum statistical models in the sense of calculating the partition function in the thermodynamic limit. One of the remarkable recent applications is their usage in the construction of the fermionic basis [21, 25, 26, 42] for the observables of the XXZ spin chain closely related to the six-vertex model.

It seems that the most productive, although not comprehensive, modern approach to the theory of quantum integrable systems is the approach based on the concept of quantum group invented by Drinfeld and Jimbo [34, 39]. In this approach, all the objects describing the model and related to its integrability originate from the universal R -matrix, and the functional relations are consequences of the properties of the appropriate representations of the quantum group. It was clearly realized by Bazhanov, Lukyanov and Zamolodchikov [14–16]. The present paper can be considered as an introduction to the application of the theory of quantum groups to formulation of integrable systems and derivation of the corresponding functional relations. We were prompted to write it by the absence of a detailed and exhaustive consideration of the method in the literature. One more reason was a desire to fix the terminology and notations for our future research.

In section 2, we discuss the original approach to formulation and investigation of quantum square lattice vertex models. We introduce basic objects, and for the case of the six-vertex model reproduce the Baxter's reasonings for the appearance of the functional relations. In section 3, all the objects describing an integrable lattice vertex model and used to integrate it are constructed starting from a quantum group. Two representations of the quantum group should be fixed to describe a model. Here, particularly by historical reasons, the corresponding representation spaces are called the auxiliary space and the quantum space. In most cases there is an associated quantum mechanical model defined in the quantum space or its tensor power. In fact, a lattice model arises when we take finite-dimensional representations, and the associated quantum mechanical model here is some spin chain. The basic example here is the six-vertex model and XXZ spin chain, see, for example, the book by Baxter [8]. If the quantum space is the representation space of certain infinite-dimensional vertex representation of the quantum group, we have a two-dimensional field theory [11, 14–16]. In section 4 we consider the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$. The full set of functional relation in the universal form independent of the choice of representation of the quantum group in the quantum space is derived in section 5.

We assume that the reader is acquainted with the basic facts on quantum groups. Beside the original papers [34, 39], we recommend for this purpose the book by Chari and Pressley [29].

Below, a vector space is a vector space over the field \mathbb{C} of complex numbers, and an algebra is a complex associative unital algebra. In fact, all general definitions, given

in section 3, can easily be extended to the case of algebras and vector spaces over a general field or even a general commutative unital ring.

The symbol ' \otimes ' is used for the tensor product of vector spaces, for the tensor product of homomorphisms and for the Kronecker product of matrices. Depending on the context, the symbol '1' means the number one, the unit of an algebra or the unit matrix. We denote by $\mathcal{L}(\mathfrak{g})$ the loop Lie algebra of a finite-dimensional simple Lie algebra \mathfrak{g} , by $\tilde{\mathcal{L}}(\mathfrak{g})$ its standard central extension, and by $\hat{\mathcal{L}}(\mathfrak{g})$ the Lie algebra obtained from $\tilde{\mathcal{L}}(\mathfrak{g})$ by adjoining the standard derivation, see, for example, the book by Kac [43].

2. SQUARE LATTICE VERTEX MODELS

2.1. Vertex models and transfer matrix. Here we recall the basic facts on integrable two-dimensional square lattice vertex models and show how functional relations arise in the case of the six-vertex model.

First of all, the models in question are quantum statistical models whose properties in the state of thermodynamic equilibrium are described by the partition function. Label by \mathcal{C} the possible eigenstates of the Hamiltonian¹ of the system under consideration and denote by $\mathcal{E}_{\mathcal{C}}$ the corresponding energy. The partition function is²

$$Z = \sum_{\mathcal{C}} \exp(-\beta \mathcal{E}_{\mathcal{C}}),$$

where $\beta = 1/k_B T$ with k_B the Boltzmann constant and T the temperature. The quantity $\exp(-\beta \mathcal{E}_{\mathcal{C}})$ is called the Boltzmann weight of the configuration \mathcal{C} .

Consider now a two-dimensional square lattice, see Figure 1, and assume that some

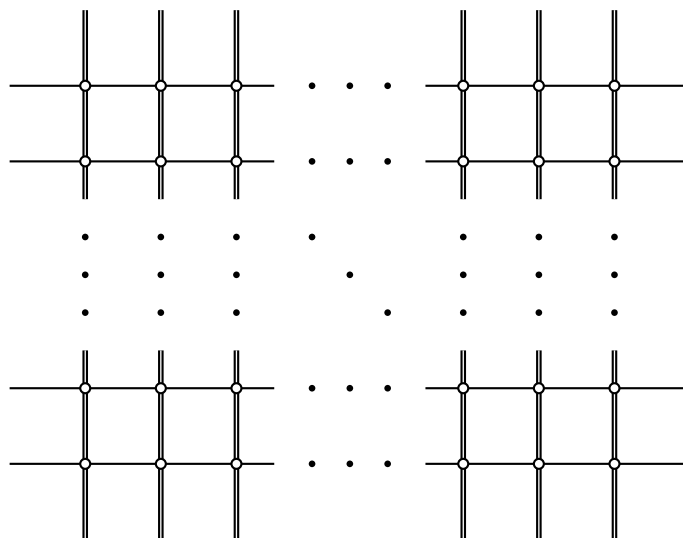


FIGURE 1. Two-dimensional square lattice.

particles are located at its vertices. Any horizontal bond of the lattice can be in one of ℓ states, and a vertical bond in one of k states. This defines a configuration of the system. Usually one also imposes boundary conditions. The simplest case here is the periodic boundary condition, where for each horizontal and vertical row the state of the first bond coincides with the state of the last bond.

¹For lattice models we usually call an eigenstate of the Hamiltonian a configuration of the system.

²We restrict ourselves to consideration of the canonical ensemble, see, for example, the book [38].

The energy of a configuration is the sum of the energies associated with the vertices. The energy $\mathcal{E}_{\mathcal{C},\mathcal{V}}$ associated with a vertex \mathcal{V} depends on the vertex itself and on the configuration \mathcal{C} . Therefore, we have

$$Z = \sum_{\mathcal{C}} \exp\left(-\beta \sum_{\mathcal{V}} \mathcal{E}_{\mathcal{C},\mathcal{V}}\right) = \sum_{\mathcal{C}} \prod_{\mathcal{V}} \exp\left(-\beta \mathcal{E}_{\mathcal{C},\mathcal{V}}\right).$$

Assume that $\mathcal{E}_{\mathcal{C},\mathcal{V}}$ depends only on the states of the bonds connecting \mathcal{V} with the neighboring vertices. We label the states of the horizontal and vertical bonds by the integers from 1 to ℓ and from 1 to k respectively, and denote the energy associated with a vertex by $\mathcal{E}_{ai|bj}$, where the indices correspond to the states of the bonds as is shown in Figure 2.

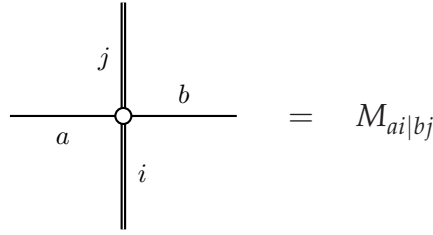


FIGURE 2. The association of the indices with the bonds.

It is convenient to introduce the Boltzmann weight of a vertex

$$M_{ai|bj} = \exp(-\beta \mathcal{E}_{ai|bj}).$$

It is clear that the Boltzmann weight of a configuration is the product of the Boltzmann weights of the vertices, and the summation over the configurations is the summation over the indices associated with the bonds. One can start with the summation over the indices associated with the horizontal bonds of a row excluding the first and the last bonds. This gives the quantities

$$M_{ai_1 i_2 \dots i_n | b j_1 j_2 \dots j_n} = \sum_{c_1, c_2, \dots, c_{n-1}} M_{ai_1 | c_1 j_1} M_{c_1 i_2 | c_2 j_2} \dots M_{c_{n-1} i_n | b j_n}, \quad (2.1)$$

where n is the number of the vertices in a horizontal row. Now we sum over the states of the remaining bonds of a horizontal row. If we assume the periodic boundary conditions, we should put in equation (2.1) $b = a$ and sum over a . More generally, one can multiply (2.1) by some quantities F_{ba} and sum over a and b independently. This can be considered as a generalization of boundary conditions called quasi-periodic or twisted. As the result we obtain the quantities

$$T_{i_1 i_2 \dots i_n | j_1 j_2 \dots j_n} = \sum_{c_1, c_2, \dots, c_{n-1}, a, b} M_{ai_1 | c_1 j_1} M_{c_1 i_2 | c_2 j_2} \dots M_{c_{n-1} i_n | b j_n} F_{ba},$$

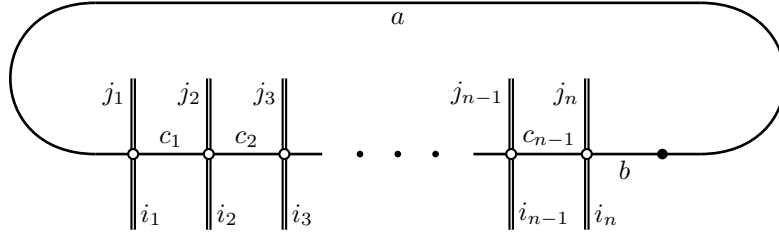
which can be graphically interpreted by Figure 3, where

$$\begin{array}{c} a \\ \bullet \\ \hline b \end{array} = F_{ba},$$

and the summation over the indices associated with the internal lines is implied. Define a $k^n \times k^n$ matrix

$$\mathbb{T} = (T_{i_1 i_2 \dots i_n | j_1 j_2 \dots j_n})$$

called the transfer matrix. It is clear that the summation over the states of the horizontal bonds of two adjacent horizontal rows and over the states of the vertical bonds between them gives the entries of the matrix \mathbb{T}^2 . Summing over the states of the horizontal bonds of all horizontal rows and over the states of the vertical bonds between


 FIGURE 3. Graphical interpretation of the quantities $T_{i_1 i_2 \dots i_n | j_1 j_2 \dots j_n}$.

them we obtain the entries of the matrix \mathbb{T}^m , where m is the number of the horizontal rows. Assuming the purely periodic boundary conditions for the vertical rows we see that the summation over the states of the remaining bonds gives the trace of this matrix. Thus, we come to the equation

$$Z = \text{tr}(\mathbb{T}^m).$$

Recall that the statistical physics describes systems of large numbers of particles. Hence, we are primarily interested in the case of large n and m . If λ_{\max} is the maximal eigenvalue of the transfer matrix \mathbb{T} and it is nondegenerate, then for large m we have the estimation

$$Z \sim \lambda_{\max}^m.$$

Therefore, the problem of calculating the partition function is reduced to the problem of finding the maximal eigenvalue of the transfer matrix for large n . In some cases it can be done using the Bethe ansatz or some its modification. In fact, the applicability of the Bethe ansatz is a manifestation of a rich algebraic structure behind the model under consideration. To reveal this structure, it is useful to introduce spectral parameters associated with the rows and columns of the lattice, see Figure 4, and assume that

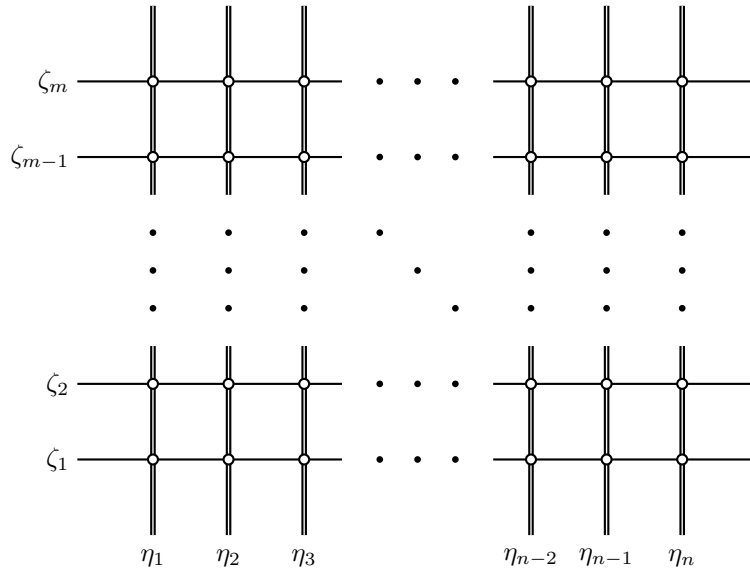


FIGURE 4. Spectral parameters.

the Boltzmann weight of a vertex depends on the corresponding row and column spectral parameters. So we write $\mathbb{T}(\zeta | \eta_1, \dots, \eta_n)$ for the transfer operator. The case $\eta_1 = \eta_2 = \dots = \eta_n = 1$ is called homogeneous.

2.2. Integrable models. The vertex model under consideration is said to be integrable if we have

$$[\mathbb{T}(\zeta_1|\eta_1, \dots, \eta_n), \mathbb{T}(\zeta_2|\eta_1, \dots, \eta_n)] = 0 \quad (2.2)$$

for any ζ_1 and ζ_2 . It follows from this equation that the transfer matrix $\mathbb{T}(\zeta|\eta_1, \dots, \eta_n)$ can be put into Jordan normal form by a similarity transformation which does not depend on ζ . One can show that equation (2.2) is valid, in particular, if there exist $\ell^2 \times \ell^2$ quantities $R_{a_1 a_2 | b_1 b_2}(\zeta_1 | \zeta_2)$ such that

$$\begin{aligned} \sum_{c_1, c_2, k} R_{a_1 a_2 | c_1 c_2}(\zeta_1 | \zeta_2) M_{c_1 i | b_1 k}(\zeta_1 | \eta) M_{c_2 k | b_2 j}(\zeta_2 | \eta) \\ = \sum_{c_1, c_2, k} M_{a_2 i | c_2 k}(\zeta_2 | \eta) M_{a_1 k | c_1 j}(\zeta_1 | \eta) R_{c_1 c_2 | b_1 b_2}(\zeta_1 | \zeta_2), \end{aligned} \quad (2.3)$$

and

$$\sum_{c_1, c_2} R_{a_1 a_2 | c_1 c_2}(\zeta_1 | \zeta_2) F_{c_1 b_1}(\zeta_1) F_{c_2 b_2}(\zeta_2) = \sum_{c_1, c_2} F_{a_1 c_1}(\zeta_1) F_{a_2 c_2}(\zeta_2) R_{c_1 c_2 | b_1 b_2}(\zeta_1 | \zeta_2). \quad (2.4)$$

The model possesses the richest algebraic structure if additionally

$$\begin{aligned} \sum_{c_1, c_2, c_3} R_{a_1 a_2 | c_1 c_2}(\zeta_1 | \zeta_2) R_{c_1 a_3 | b_1 c_3}(\zeta_1 | \zeta_3) R_{c_2 c_3 | b_2 b_3}(\zeta_2 | \zeta_3) \\ = \sum_{c_1, c_2, c_3} R_{a_2 a_3 | c_2 c_3}(\zeta_2 | \zeta_3) R_{a_1 c_3 | c_1 b_3}(\zeta_1 | \zeta_3) R_{c_1 c_2 | b_1 b_2}(\zeta_1 | \zeta_2). \end{aligned} \quad (2.5)$$

This is the famous Yang-Baxter equation.

2.3. Functional relations for the six-vertex model. The standard example of a quantum statistical vertex model is the six-vertex model. Here any horizontal and vertical bond can be in one of two states labelled by 1 and 2, and the Boltzmann weights $M_{ai|bj}(\zeta|\eta)$ can be arranged into the matrix

$$(M_{ai|bj}(\zeta|\eta)) = \begin{pmatrix} a(\zeta\eta^{-1}) & 0 & 0 & 0 \\ 0 & b(\zeta\eta^{-1}) & c(\zeta\eta^{-1}) & 0 \\ 0 & c(\zeta\eta^{-1}) & b(\zeta\eta^{-1}) & 0 \\ 0 & 0 & 0 & a(\zeta\eta^{-1}) \end{pmatrix}, \quad (2.6)$$

where

$$a(\zeta) = q\zeta - q^{-1}\zeta^{-1}, \quad b(\zeta) = \zeta - \zeta^{-1}, \quad c(\zeta) = q - q^{-1}, \quad (2.7)$$

and we order the pairs of indices as 11, 12, 21, 22. The parameter q is a fixed nonzero complex number. We see that the Boltzmann weights are different from zero only for six vertex configurations, hence the name of the model. Equation (2.3) is satisfied with

$$R_{a_1 a_2 | b_1 b_2}(\zeta_1 | \zeta_2) = M_{a_1 a_2 | b_1 b_2}(\zeta_1 | \zeta_2). \quad (2.8)$$

The usual choice for $F_{ab}(\zeta)$ is

$$F_{11}(\zeta) = q^\phi, \quad F_{12}(\zeta) = 0, \quad F_{21}(\zeta) = 0, \quad F_{22}(\zeta) = q^{-\phi},$$

where ϕ is an arbitrary complex number. One can verify that (2.4) is satisfied.

To find eigenvectors and eigenvalues of the transfer matrix it is convenient to use the algebraic Bethe ansatz [59, 61]. This approach shows that there are the eigenvectors of the transfer matrix with the eigenvalues

$$\lambda(\zeta|\eta_1, \dots, \eta_n) = q^\phi \prod_{i=1}^n a(\zeta\eta_i^{-1}) \prod_{\ell=1}^p \frac{a(\zeta_\ell \zeta^{-1})}{b(\zeta_\ell \zeta^{-1})} + q^{-\phi} \prod_{i=1}^n b(\zeta\eta_i^{-1}) \prod_{\ell=1}^p \frac{a(\zeta_\ell \zeta_\ell^{-1})}{b(\zeta_\ell \zeta_\ell^{-1})}, \quad (2.9)$$

where $0 \leq p \leq n$, and ζ_1, \dots, ζ_p satisfy the Bethe equations

$$q^{2\phi} \prod_{i=1}^n \frac{a(\zeta_m \eta_i^{-1})}{b(\zeta_m \eta_i^{-1})} = (-1)^{p+1} \prod_{\ell=1}^p \frac{a(\zeta_m \zeta_\ell^{-1})}{a(\zeta_\ell \zeta_m^{-1})}, \quad m = 1, \dots, p.$$

It can be shown that all eigenvalues can be obtained by the Bethe ansatz, and that the corresponding eigenvectors form a basis of \mathbb{C}^{2^n} , see, for example, [9] and references therein. It is important that these eigenvectors do not depend on ζ , that is in fact a consequence of equation (2.2). Now it is clear that there is a matrix $\mathbb{O}(\eta_1, \dots, \eta_n)$ such that

$$\mathbb{O}(\eta_1, \dots, \eta_n) \mathbb{T}_d(\zeta | \eta_1, \dots, \eta_n) \mathbb{O}^{-1}(\eta_1, \dots, \eta_n) = \mathbb{T}(\zeta | \eta_1, \dots, \eta_n), \quad (2.10)$$

where $\mathbb{T}_d(\zeta | \eta_1, \dots, \eta_n)$ is a diagonal matrix.

For a given solution of the Bethe equations we define the function

$$\theta(\zeta | \eta_1, \dots, \eta_n) = \prod_{\ell=1}^p b(\zeta_\ell(\eta_1, \dots, \eta_n) \zeta^{-1}), \quad (2.11)$$

where the dependence of the Bethe roots ζ_1, \dots, ζ_p on the spectral parameters η_1, \dots, η_n is shown explicitly. Now we can rewrite relation (2.9) as

$$\begin{aligned} & \lambda(\zeta | \eta_1, \dots, \eta_n) \theta(\zeta | \eta_1, \dots, \eta_n) \\ &= q^\phi \prod_{i=1}^n a(\zeta \eta_i^{-1}) \theta(q^{-1} \zeta | \eta_1, \dots, \eta_n) + q^{-\phi} \prod_{i=1}^n b(\zeta \eta_i^{-1}) \theta(q \zeta | \eta_1, \dots, \eta_n). \end{aligned} \quad (2.12)$$

The matrix $\mathbb{T}_d(\zeta | \eta_1, \dots, \eta_n)$ in (2.10) is a diagonal matrix with entries being the eigenvalues of $\mathbb{T}(\zeta | \eta_1, \dots, \eta_n)$ of the form (2.9). Denote by $\mathbb{Q}_d(\zeta | \eta_1, \dots, \eta_n)$ the diagonal matrix whose entries are the corresponding functions $\theta(\zeta | \eta_1, \dots, \eta_n)$ given by (2.11). It follows from (2.12) that

$$\begin{aligned} & \mathbb{T}_d(\zeta | \eta_1, \dots, \eta_n) \mathbb{Q}_d(\zeta | \eta_1, \dots, \eta_n) \\ &= q^\phi \prod_{i=1}^n a(\zeta \eta_i^{-1}) \mathbb{Q}_d(q^{-1} \zeta | \eta_1, \dots, \eta_n) + q^{-\phi} \prod_{i=1}^n b(\zeta \eta_i^{-1}) \mathbb{Q}_d(q \zeta | \eta_1, \dots, \eta_n). \end{aligned} \quad (2.13)$$

Define the matrix

$$\mathbb{Q}(\zeta | \eta_1, \dots, \eta_n) = \mathbb{O}(\eta_1, \dots, \eta_n) \mathbb{Q}_d(\zeta | \eta_1, \dots, \eta_n) \mathbb{O}^{-1}(\eta_1, \dots, \eta_n).$$

Since the matrix $\mathbb{Q}_d(\zeta | \eta_1, \dots, \eta_n)$ does not depend on ζ , it follows from (2.13) that

$$\begin{aligned} & \mathbb{T}(\zeta | \eta_1, \dots, \eta_n) \mathbb{Q}(\zeta | \eta_1, \dots, \eta_n) \\ &= q^\phi \prod_{i=1}^n a(\zeta \eta_i^{-1}) \mathbb{Q}(q^{-1} \zeta | \eta_1, \dots, \eta_n) + q^{-\phi} \prod_{i=1}^n b(\zeta \eta_i^{-1}) \mathbb{Q}(q \zeta | \eta_1, \dots, \eta_n). \end{aligned} \quad (2.14)$$

This functional equation is called the Baxter's TQ -equation. By construction, we also have

$$[\mathbb{Q}(\zeta_1 | \eta_1, \dots, \eta_n), \mathbb{T}(\zeta_2 | \eta_1, \dots, \eta_n)] = 0, \quad (2.15)$$

$$[\mathbb{Q}(\zeta_1 | \eta_1, \dots, \eta_n), \mathbb{Q}(\zeta_2 | \eta_1, \dots, \eta_n)] = 0 \quad (2.16)$$

for any ζ_1 and ζ_2 . We call (2.2), (2.15), (2.16) and (2.14) functional relations. They are equivalent to the Bethe ansatz in the sense that they can be used to find the eigenvalues of the transfer matrix, see, for example, the book [8].

In the next section we explain how the objects necessary for the integration of an integrable model are related to its background algebraic structure.

3. OBJECTS DEFINED BY THE UNIVERSAL R -MATRIX

In this section A is a \mathbb{Z} -graded quasitriangular Hopf algebra over the field \mathbb{C} with the comultiplication Δ and the universal R -matrix \mathcal{R} . Some relevant definitions are reproduced in appendices **C** and **D**.

3.1. R -operators. Let φ be a representation of the algebra A in the vector space V .³ Given $\nu \in \mathbb{C}^\times$, we denote

$$\varphi_\nu = \varphi \circ \Phi_\nu, \quad (3.1)$$

where the mapping Φ_ν is defined by relation (D.1). For any $\zeta_1, \zeta_2 \in \mathbb{C}^\times$ we define

$$R_\varphi(\zeta_1|\zeta_2) = (\varphi_{\zeta_1} \otimes \varphi_{\zeta_2})(\mathcal{R}), \quad (3.2)$$

where \mathcal{R} is the universal R -matrix of A . Having in mind the relation to integrable systems, we call ζ_1 and ζ_2 *spectral parameters*. It is clear that $R_\varphi(\zeta_1|\zeta_2)$ is an element of $\text{End}(V) \otimes \text{End}(V) \cong \text{End}(V \otimes V)$. We call it an R -operator.

It appears often that the universal R -matrix \mathcal{R} of A satisfies the equation

$$(\Phi_\nu \otimes \Phi_\nu)(\mathcal{R}) = \mathcal{R} \quad (3.3)$$

for any $\nu \in \mathbb{C}^\times$. From the point of view of the natural \mathbb{Z} -gradation of $A \otimes A$, induced by the \mathbb{Z} -gradation of A , this means that the universal R -matrix \mathcal{R} belongs to the zero grade subalgebra $(A \otimes A)_0$, see appendix **D**. In this case, using the equation,

$$\varphi_{\zeta\nu} \stackrel{(3.1)}{=} \varphi \circ \Phi_{\zeta\nu} \stackrel{(D.2)}{=} \varphi_\zeta \circ \Phi_\nu, \quad (3.4)$$

we obtain

$$\begin{aligned} R_\varphi(\zeta_1\nu|\zeta_2\nu) &\stackrel{(3.2)}{=} (\varphi_{\zeta_1\nu} \otimes \varphi_{\zeta_2\nu})(\mathcal{R}) \\ &\stackrel{(3.4)}{=} ((\varphi_{\zeta_1} \otimes \varphi_{\zeta_2}) \circ (\Phi_\nu \otimes \Phi_\nu))(\mathcal{R}) \stackrel{(3.3)}{=} (\varphi_{\zeta_1} \otimes \varphi_{\zeta_2})(\mathcal{R}) \stackrel{(3.2)}{=} R_\varphi(\zeta_1|\zeta_2). \end{aligned} \quad (3.5)$$

Thus, $R_\varphi(\zeta_1|\zeta_2)$ depends only on the combination $\zeta_1\zeta_2^{-1}$, and one can introduce the R -operator

$$R_\varphi(\zeta) = R_\varphi(\zeta|1)$$

which depends on only one spectral parameter and determines the R -operator depending on two spectral parameters, via the equation

$$R_\varphi(\zeta_1|\zeta_2) = R_\varphi(\zeta_1\zeta_2^{-1}).$$

Return to a general situation and apply the mapping $\varphi_{\zeta_1} \otimes \varphi_{\zeta_2} \otimes \varphi_{\zeta_3}$ to both sides of the Yang-Baxter equation (C.5) for the universal R -matrix. We obtain the Yang-Baxter equation for the R -operator,

$$R_\varphi^{12}(\zeta_1|\zeta_2) R_\varphi^{13}(\zeta_1|\zeta_3) R_\varphi^{23}(\zeta_2|\zeta_3) = R_\varphi^{23}(\zeta_2|\zeta_3) R_\varphi^{13}(\zeta_1|\zeta_3) R_\varphi^{12}(\zeta_1|\zeta_2). \quad (3.6)$$

In the case where equation (3.3) is valid, for the R -operator depending on one spectral parameter we have

$$R_\varphi^{12}(\zeta_{12}) R_\varphi^{13}(\zeta_{13}) R_\varphi^{23}(\zeta_{23}) = R_\varphi^{23}(\zeta_{23}) R_\varphi^{13}(\zeta_{13}) R_\varphi^{12}(\zeta_{12}).$$

Here and below we denote $\zeta_{ij} = \zeta_i\zeta_j^{-1}$.

One often uses two operators directly related to the R -operator defined by equation (3.2). One of them is defined as

$$\check{R}_\varphi(\zeta_1|\zeta_2) = PR_\varphi(\zeta_1|\zeta_2),$$

³For the case of square lattice models the vector space V is the auxiliary space.

where P is the permutation operator in $V \otimes V$, see appendix B. Using this definition, we can rewrite the left hand side of the Yang–Baxter equation in the following way:

$$\begin{aligned} R_\varphi^{12}(\zeta_1|\zeta_2)R_\varphi^{13}(\zeta_1|\zeta_3)R_\varphi^{23}(\zeta_2|\zeta_3) \\ = P^{12}\check{R}_\varphi^{12}(\zeta_1|\zeta_2)P^{13}\check{R}_\varphi^{13}(\zeta_1|\zeta_3)P^{23}\check{R}_\varphi^{23}(\zeta_2|\zeta_3) \\ \stackrel{(B.3)}{=} P^{12}P^{13}P^{23}\check{R}_\varphi^{23}(\zeta_1|\zeta_2)\check{R}_\varphi^{12}(\zeta_1|\zeta_3)\check{R}_\varphi^{23}(\zeta_2|\zeta_3). \end{aligned}$$

Similarly, we rewrite the right hand side as

$$R_\varphi^{23}(\zeta_2|\zeta_3)R_\varphi^{13}(\zeta_1|\zeta_3)R_\varphi^{12}(\zeta_1|\zeta_2) = P^{23}P^{13}P^{12}\check{R}_\varphi^{12}(\zeta_2|\zeta_3)\check{R}_\varphi^{23}(\zeta_1|\zeta_3)\check{R}_\varphi^{12}(\zeta_1|\zeta_2).$$

It is not difficult to verify that

$$P^{12}P^{13}P^{23} = P^{23}P^{13}P^{12},$$

therefore, the Yang–Baxter equation (3.6) is equivalent to the equation

$$\check{R}_\varphi^{23}(\zeta_1|\zeta_2)\check{R}_\varphi^{12}(\zeta_1|\zeta_3)\check{R}_\varphi^{23}(\zeta_2|\zeta_3) = \check{R}_\varphi^{12}(\zeta_2|\zeta_3)\check{R}_\varphi^{23}(\zeta_1|\zeta_3)\check{R}_\varphi^{12}(\zeta_1|\zeta_2).$$

This equation can also be written as

$$\begin{aligned} (\text{id} \otimes \check{R}_\varphi(\zeta_1|\zeta_2))(\check{R}_\varphi(\zeta_1|\zeta_3) \otimes \text{id})(\text{id} \otimes \check{R}_\varphi(\zeta_2|\zeta_3)) \\ = (\check{R}_\varphi(\zeta_2|\zeta_3) \otimes \text{id})(\text{id} \otimes \check{R}_\varphi(\zeta_1|\zeta_3))(\check{R}_\varphi(\zeta_1|\zeta_2) \otimes \text{id}). \end{aligned}$$

Another companion for the R -operator is defined as

$$\hat{R}_\varphi(\zeta_1|\zeta_2) = R_\varphi(\zeta_1|\zeta_2)P,$$

Here the Yang–Baxter equation takes the form

$$\hat{R}_\varphi^{12}(\zeta_1|\zeta_2)\hat{R}_\varphi^{23}(\zeta_1|\zeta_3)\hat{R}_\varphi^{12}(\zeta_2|\zeta_3) = \hat{R}_\varphi^{23}(\zeta_2|\zeta_3)\hat{R}_\varphi^{12}(\zeta_1|\zeta_3)\hat{R}_\varphi^{23}(\zeta_1|\zeta_2),$$

or, equivalently,

$$\begin{aligned} (\hat{R}_\varphi(\zeta_1|\zeta_2) \otimes \text{id})(\text{id} \otimes \hat{R}_\varphi(\zeta_1|\zeta_3))(\hat{R}_\varphi(\zeta_2|\zeta_3) \otimes \text{id}) \\ = (\text{id} \otimes \hat{R}_\varphi(\zeta_2|\zeta_3))(\hat{R}_\varphi(\zeta_1|\zeta_3) \otimes \text{id})(\text{id} \otimes \hat{R}_\varphi(\zeta_1|\zeta_2)). \end{aligned}$$

Assume now that the vector space V is finite-dimensional of dimension ℓ . Let $\{e_a\}$ be a basis of V , and $\{E_{ab}\}$ the corresponding basis of $\text{End}(V)$, see appendix A. We have

$$R_\varphi(\zeta_1|\zeta_2) = \sum_{a,b,c,d} E_{ab|cd} R_{ab|cd}(\zeta_1|\zeta_2) \stackrel{(A.2)}{=} \sum_{a,b,c,d} E_{ac} \otimes E_{bd} R_{ab|cd}(\zeta_1|\zeta_2),$$

where $R_{ab|cd}(\zeta_1|\zeta_2)$ are some unique complex numbers. One can verify that the Yang–Baxter equation (3.6) in terms of the quantities $R_{ab|cd}(\zeta_1|\zeta_2)$ has the form (2.5). The $\ell^2 \times \ell^2$ matrix with the entries $R_{ab|cd}(\zeta_1|\zeta_2)$ is called an R -matrix. We denote it $\mathbf{R}_\varphi(\zeta_1|\zeta_2)$.

It is not difficult to convince oneself that

$$P = \sum_{a,b} E_{ab} \otimes E_{ba}.$$

Now, defining the quantities $\check{R}_{ab|cd}(\zeta_1|\zeta_2)$ by

$$\check{R}_\varphi(\zeta_1|\zeta_2) = \sum_{a,b,c,d} E_{ac} \otimes E_{bd} \check{R}_{ab|cd}(\zeta_1|\zeta_2),$$

we see that

$$\check{R}_{ab|cd}(\zeta_1|\zeta_2) = R_{ba|cd}(\zeta_1|\zeta_2). \quad (3.7)$$

Similarly, defining the quantities $\hat{R}_{ab|cd}(\zeta_1|\zeta_2)$ by

$$\hat{R}_\varphi(\zeta_1|\zeta_2) = \sum_{a,b,c,d} E_{ac} \otimes E_{bd} \hat{R}_{ab|cd}(\zeta_1|\zeta_2),$$

we see that

$$\hat{R}_{ab|cd}(\zeta_1|\zeta_2) = R_{ab|dc}(\zeta_1|\zeta_2).$$

We denote the matrices with the entries $\check{R}_{ab|cd}(\zeta_1|\zeta_2)$ and $\hat{R}_{ab|cd}(\zeta_1|\zeta_2)$ by $\check{R}_\varphi(\zeta_1|\zeta_2)$ and $\hat{R}_\varphi(\zeta_1|\zeta_2)$ respectively.

One can also define an R -operator using two different representations, say φ_1 and φ_2 . In this case we use the notation

$$R_{\varphi_1, \varphi_2}(\zeta_1|\zeta_2) = (\varphi_{1\zeta_1} \otimes \varphi_{2\zeta_2})(\mathcal{R}).$$

In the case when φ_1 and φ_2 are representations of A in vector spaces V_1 and V_2 , respectively, the R -operator $\check{R}_{\varphi_1, \varphi_2}(\zeta_1|\zeta_2)$ serves as the intertwiner for the representations $\varphi_{1\zeta_1} \otimes_\Delta \varphi_{2\zeta_2}$ and $\varphi_{2\zeta_2} \otimes_\Delta \varphi_{1\zeta_1}$ of A in the vector spaces $V_1 \otimes V_2$ and $V_2 \otimes V_1$ respectively.⁴ To prove the intertwiner property of $\check{R}_{\varphi_1, \varphi_2}(\zeta)$ we note that

$$\begin{aligned} \varphi_{2\zeta_2} \otimes_\Delta \varphi_{1\zeta_1} &= (\varphi_{2\zeta_2} \otimes \varphi_{1\zeta_1}) \circ \Delta \\ &\stackrel{(B.4)}{=} \Pi \circ (\varphi_{1\zeta_1} \otimes \varphi_{2\zeta_2}) \circ \Pi \circ \Delta \stackrel{(C.1)}{=} \Pi \circ (\varphi_{1\zeta_1} \otimes \varphi_{2\zeta_2}) \circ \Delta^{\text{op}}. \end{aligned}$$

Hence, one can write

$$\begin{aligned} (\varphi_{2\zeta_2} \otimes_\Delta \varphi_{1\zeta_1})(a) &\stackrel{(C.2)}{=} (\Pi \circ (\varphi_{1\zeta_1} \otimes \varphi_{2\zeta_2}))(\mathcal{R} \Delta(a) \mathcal{R}^{-1}) \\ &= \Pi((\varphi_{1\zeta_1} \otimes \varphi_{2\zeta_2})(\mathcal{R})(\varphi_{1\zeta_1} \otimes_\Delta \varphi_{2\zeta_2})(a)(\varphi_{1\zeta_1} \otimes \varphi_{2\zeta_2})(\mathcal{R}^{-1})) \\ &\stackrel{(B.2)}{=} PR_{\varphi_1, \varphi_2}(\zeta_1|\zeta_2)((\varphi_{1\zeta_1} \otimes_\Delta \varphi_{2\zeta_2})(a))(R_{\varphi_1, \varphi_2}(\zeta_1|\zeta_2))^{-1}P^{-1}. \end{aligned}$$

Finally we come to the declared result,

$$\check{R}_{\varphi_1, \varphi_2}(\zeta_1|\zeta_2)(\varphi_{1\zeta_1} \otimes_\Delta \varphi_{2\zeta_2})(a) = (\varphi_{2\zeta_2} \otimes_\Delta \varphi_{1\zeta_1})(a)\check{R}_{\varphi_1, \varphi_2}(\zeta_1|\zeta_2). \quad (3.8)$$

An explicit form of R -matrices was obtained from the corresponding universal R -matrices for certain representations of the quantum groups $U_q(\mathcal{L}(\mathfrak{sl}_2))$ [19, 27, 28, 44, 51, 65], $U_q(\mathcal{L}(\mathfrak{sl}_3))$ [19, 27, 28, 65] and $U_q(\mathcal{L}(\mathfrak{sl}_3, \mu))$ [20, 44], where μ is the standard diagram automorphism of \mathfrak{sl}_3 of order 2. In fact, up to a scalar factor they coincide with the R -matrices obtained by other methods. Nevertheless, it is very useful to understand that they can be obtained from the universal R -matrices because this allows one to relate them to other objects involved into the integration process.

3.2. Monodromy operators.

3.2.1. Universal monodromy operator. Let again φ be a representation of A in the vector space V . Given $\zeta \in \mathbb{C}^\times$, we define the *universal monodromy operator* $\mathcal{M}_\varphi(\zeta)$ by the equation

$$\mathcal{M}_\varphi(\zeta) = (\varphi_\zeta \otimes \text{id})(\mathcal{R}),$$

where the mapping φ_ζ is defined by equation (3.1). It is clear that $\mathcal{M}_\varphi(\zeta)$ is an element of the algebra $\text{End}(V) \otimes A$.

⁴We use the notation \otimes_Δ to distinguish between the tensor product of representations and the usual tensor product of mappings, so that $\varphi \otimes_\Delta \psi = (\varphi \otimes \psi) \circ \Delta$.

Applying the mapping $\varphi_{\zeta_1} \otimes \varphi_{\zeta_2} \otimes \text{id}$ to both sides of the Yang–Baxter equation (C.5), we obtain the equation

$$R_\varphi^{12}(\zeta_1|\zeta_2)\mathcal{M}_\varphi^{13}(\zeta_1)\mathcal{M}_\varphi^{23}(\zeta_2) = \mathcal{M}_\varphi^{23}(\zeta_2)\mathcal{M}_\varphi^{13}(\zeta_1)R_\varphi^{12}(\zeta_1|\zeta_2).$$

Multiply both sides of the above equation by $P^{12} \otimes 1$ and use equation (B.3). This gives

$$\check{R}_\varphi^{12}(\zeta_1|\zeta_2)\mathcal{M}_\varphi^{13}(\zeta_1)\mathcal{M}_\varphi^{23}(\zeta_2) = \mathcal{M}_\varphi^{13}(\zeta_2)\mathcal{M}_\varphi^{23}(\zeta_1)\check{R}_\varphi^{12}(\zeta_1|\zeta_2). \quad (3.9)$$

There is a matrix equivalent of this equation.

Assume that the vector space V is finite-dimensional of dimension ℓ . Let $\{e_a\}$ be a basis of V and $\{E_{ab}\}$ the corresponding basis of $\text{End}(V)$, see appendix A. One can write

$$\mathcal{M}_\varphi(\zeta) = \sum_{a,b} E_{ab} \otimes \mathcal{M}_{ab}(\zeta),$$

where $\mathcal{M}_{ab}(\zeta)$ are some unique elements of the algebra A . Denote by $\mathbf{M}_\varphi(\zeta)$ the matrix with the entries $\mathcal{M}_{ab}(\zeta)$. The matrix $\mathbf{M}_\varphi(\zeta)$ is an element of $\text{Mat}_\ell(A)$. We call it a *universal monodromy matrix*. Now, it follows from (3.9) that

$$\check{R}_\varphi(\zeta_1|\zeta_2)(\mathbf{M}_\varphi(\zeta_1) \boxtimes \mathbf{M}_\varphi(\zeta_2)) = (\mathbf{M}_\varphi(\zeta_2) \boxtimes \mathbf{M}_\varphi(\zeta_1))\check{R}_\varphi(\zeta_1|\zeta_2). \quad (3.10)$$

Here the operation \boxtimes is defined by equation (A.4), and, using the canonical embedding of the field \mathbb{C} into A , we treat $\check{R}_\varphi(\zeta)$ as an element of $\text{Mat}_{\ell^2}(A)$. It is worth to remind here that \boxtimes is a natural generalization of the Kronecker product to the case of matrices with noncommuting entries.

3.2.2. Monodromy operator. Let φ and ψ be representations of A in the vector spaces V and U respectively.⁵ Given $\zeta, \eta \in \mathbb{C}^\times$, we define the *monodromy operator* $M_{\varphi,\psi}(\zeta|\eta)$ by the equation

$$M_{\varphi,\psi}(\zeta|\eta) = (\varphi_\zeta \otimes \psi_\eta)(\mathcal{R}),$$

where the mapping φ_ζ is defined by equation (3.1) and the mapping ψ_η is defined in the similar way. It is clear that $M_{\varphi,\psi}(\zeta|\eta)$ is an element of $\text{End}(V) \otimes \text{End}(U)$.

One should note that the monodromy operator $M_{\varphi,\psi}(\zeta|\eta)$ coincides with the R -operator $R_{\varphi,\psi}(\zeta|\eta)$. Nevertheless, we use different names due to different roles these objects play in the integration process.

Since A is a bialgebra, one can also define the monodromy operator⁶

$$M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = (\varphi_\zeta \otimes (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n}))(\mathcal{R}),$$

where η_1, \dots, η_n are some nonzero complex numbers. Note that this monodromy operator is an element of $\text{End}(V) \otimes \text{End}(U)^{\otimes n} \cong \text{End}(V) \otimes \text{End}(U^{\otimes n})$. In fact, one can use different representations, say ψ_1, \dots, ψ_n , for different factors of the tensor product. This is the case for the construction of the quantum transfer matrix [50] and for the description of integrable defects [2, 30, 31, 64].

For the opposite comultiplication we have

$$(\text{id} \otimes \Delta^{\text{op}})(\mathcal{R}) \stackrel{(C.1)}{=} (\text{id} \otimes \Pi^{23})(\text{id} \otimes \Delta)(\mathcal{R}) \stackrel{(C.4)}{=} (\text{id} \otimes \Pi^{23})(\mathcal{R}^{13}\mathcal{R}^{12}) \stackrel{(B.1)}{=} \mathcal{R}^{12}\mathcal{R}^{13}.$$

Therefore, we can see that

$$M_{\varphi,\psi}(\zeta|\eta_1, \eta_2) = (\varphi_\zeta \otimes (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \psi_{\eta_2}))(\mathcal{R})$$

⁵For the case of square lattice models the vector space U is the quantum space.

⁶We use the comultiplication Δ^{op} instead of Δ to have relations similar to those which usually arise for integrable systems.

$$= ((\varphi_\zeta \otimes \psi_{\eta_1} \otimes \psi_{\eta_2}) \circ (\text{id} \otimes \Delta^{\text{op}}))(\mathcal{R}) = M_{\varphi,\psi}^{12}(\zeta|\eta_1)M_{\varphi,\psi}^{13}(\zeta|\eta_2).$$

In general, we have

$$M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = M_{\varphi,\psi}^{12}(\zeta|\eta_1) \dots M_{\varphi,\psi}^{1,n+1}(\zeta|\eta_n). \quad (3.11)$$

One often labels the first factor of the tensor product $V \otimes U^{\otimes n}$ by 0 and the rest by 1, \dots, n . In this case the above relation takes the form

$$M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = M_{\varphi,\psi}^{01}(\zeta|\eta_1) \dots M_{\varphi,\psi}^{0n}(\zeta|\eta_n).$$

If equation (3.3) is satisfied, in the same way as for the case of R -operators, see (3.5), we obtain

$$M_{\varphi,\psi}(\zeta\nu|\eta\nu) = M_{\varphi,\psi}(\zeta|\eta). \quad (3.12)$$

Therefore, we can write

$$M_{\varphi,\psi}(\zeta|\eta) = M_{\varphi,\psi}(\zeta\eta^{-1}),$$

where $M_{\varphi,\psi}(\zeta) = M_{\varphi,\psi}(\zeta|1)$. Furthermore, in this case equation (3.11) gives

$$M_{\varphi,\psi}(\zeta\nu|\eta_1\nu, \dots, \eta_n\nu) = M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n)$$

for any $\nu \in \mathbb{C}^\times$.

Assume now that the vector space V is finite-dimensional, $\{e_a\}$ is a basis of V , and $\{E_{ab}\}$ the corresponding basis of $\text{End}(V)$. Represent the monodromy operator as

$$M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = \sum_{a,b} E_{ab} \otimes M_{ab}(\zeta|\eta_1, \dots, \eta_n), \quad (3.13)$$

where $M_{ab}(\zeta|\eta_1, \dots, \eta_n)$ are elements of $\text{End}(U)^{\otimes n} \cong \text{End}(U^{\otimes n})$. It is clear that

$$M_{ab}(\zeta|\eta_1, \dots, \eta_n) = (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n})(\mathcal{M}_{ab}(\zeta)), \quad (3.14)$$

where $\mathcal{M}_{ab}(\zeta)$ are the entries of the universal monodromy matrix defined in section 3.2.1. Denote by $\mathbf{M}_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n)$ the matrix with the entries $M_{ab}(\zeta|\eta_1, \dots, \eta_n)$. It is an element of $\text{Mat}_\ell(\text{End}(U)^{\otimes n}) \cong \text{Mat}_\ell(\text{End}(U^{\otimes n}))$. Using (3.11), one can show that

$$\mathbf{M}_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = \mathbf{M}_{\varphi,\psi}(\zeta|\eta_1) \square \dots \square \mathbf{M}_{\varphi,\psi}(\zeta|\eta_n), \quad (3.15)$$

where the operation \square is defined by (A.3). Applying the mapping $\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n}$ to the entries of matrices in both sides of equation (3.10) and taking into account relation (3.14), we see that

$$\begin{aligned} \check{\mathbf{R}}_\varphi(\zeta_1|\zeta_2)(\mathbf{M}_{\varphi,\psi}(\zeta_1|\eta_1, \dots, \eta_n) \boxtimes \mathbf{M}_{\varphi,\psi}(\zeta_2|\eta_1, \dots, \eta_n)) \\ = (\mathbf{M}_{\varphi,\psi}(\zeta_2|\eta_1, \dots, \eta_n) \boxtimes \mathbf{M}_{\varphi,\psi}(\zeta_1|\eta_1, \dots, \eta_n)) \check{\mathbf{R}}_\varphi(\zeta_1|\zeta_2). \end{aligned}$$

Relations of this type are the basis of the algebraic Bethe ansatz [59, 61].

Now assume that the vector space U is finite-dimensional. Let $\{e_i\}$ be a basis of U , and $\{E_{ij}\}$ the corresponding basis of $\text{End}(U)$. Represent the monodromy operator as

$$M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = \sum_{\substack{i_1, \dots, i_n \\ j_1, \dots, j_n}} M_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n) \otimes E_{i_1 j_1} \otimes \dots \otimes E_{i_n j_n}, \quad (3.16)$$

where $M_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n)$ are elements of $\text{End}(V)$. Introducing the matrix

$$\mathbb{M}_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = (M_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n))$$

and using (3.11), we obtain the equation

$$\mathbb{M}_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = \mathbb{M}_{\varphi,\psi}(\zeta|\eta_1) \boxtimes \dots \boxtimes \mathbb{M}_{\varphi,\psi}(\zeta|\eta_n). \quad (3.17)$$

If both vector spaces V and U are finite-dimensional, we can write

$$M_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = \sum_{\substack{a,i_1,\dots,i_n \\ b,j_1,\dots,j_n}} M_{ai_1\dots i_n|bj_1\dots j_n}(\zeta|\eta_1, \dots, \eta_n) E_{ab} \otimes E_{i_1j_1} \otimes \dots \otimes E_{i_nj_n},$$

where $M_{ai_1\dots i_n|bj_1\dots j_n}(\zeta|\eta_1, \dots, \eta_n)$ are elements of the field \mathbb{C} . Here relation (3.11) gives

$$\begin{aligned} M_{ai_1i_2\dots i_n|bj_1j_2\dots j_n}(\zeta|\eta_1, \dots, \eta_n) \\ = \sum_{c_1,c_2,\dots,c_{n-1}} M_{ai_1|c_1j_1}(\zeta|\eta_1) M_{c_1i_2|c_2j_2}(\zeta|\eta_2) \dots M_{c_{n-1}i_n|bj_n}(\zeta|\eta_n). \end{aligned}$$

This is equation (2.1) with the dependence on the spectral parameters included.

For usual square lattice models, such as the six-vertex model, the representation ψ coincides with the representation φ . Here the monodromy operator $M_{\varphi,\psi}(\zeta|\eta) = M_{\varphi,\varphi}(\zeta|\eta)$ coincides with the corresponding R -operator $R_{\varphi}(\zeta|\eta)$.

3.3. Transfer operators. The transfer operators are obtained via taking the trace over the representation space V of the representation φ used to define the monodromy operators. Some necessary information on traces can be found in appendix E.

3.3.1. Universal transfer operator. Let φ be a representation of the algebra A in the vector space V , and t a group-like element of A ,

$$\Delta(t) = t \otimes t. \quad (3.18)$$

We define the *universal transfer operator* as

$$\mathcal{T}_{\varphi}(\zeta) = (\mathrm{tr}_V \otimes \mathrm{id})(\mathcal{M}_{\varphi}(\zeta)(\varphi_{\zeta}(t) \otimes 1)) = ((\mathrm{tr}_V \circ \varphi_{\zeta}) \otimes \mathrm{id})(\mathcal{R}(t \otimes 1)),$$

where the mapping φ_{ζ} is defined by relation (3.1). We call t a *twist element*.

It is easy to see that

$$\Delta^{\mathrm{op}}(t) \stackrel{(C.1)}{=} \Pi(\Delta(t)) \stackrel{(3.18)}{=} t \otimes t.$$

From the other hand

$$\Delta^{\mathrm{op}}(t) \stackrel{(C.2)}{=} \mathcal{R} \Delta(t) \mathcal{R}^{-1} \stackrel{(3.18)}{=} \mathcal{R}(t \otimes t) \mathcal{R}^{-1}.$$

Therefore, we have the equation

$$\mathcal{R}(t \otimes t) = (t \otimes t) \mathcal{R}. \quad (3.19)$$

The above equation can be written as

$$\mathcal{R}^{12} t^1 t^2 = t^1 t^2 \mathcal{R}^{12}.$$

Multiplying the Yang–Baxter equation (C.5) from the right by $t^1 t^2$ and using the above equation, we obtain

$$(\mathcal{R}^{13} t^1)(\mathcal{R}^{23} t^2) = (\mathcal{R}^{12})^{-1} (\mathcal{R}^{23} t^2) (\mathcal{R}^{13} t^1) \mathcal{R}^{12}. \quad (3.20)$$

Applying to both sides of this equation the mapping $(\mathrm{tr} \circ \varphi_{\zeta_1}) \otimes (\mathrm{tr} \circ \varphi_{\zeta_2}) \otimes \mathrm{id}$, we come to the equation

$$\mathcal{T}_{\varphi}(\zeta_1) \mathcal{T}_{\varphi}(\zeta_2) = \mathcal{T}_{\varphi}(\zeta_2) \mathcal{T}_{\varphi}(\zeta_1). \quad (3.21)$$

Here we use equation (E.5). More generally, if φ_1 and φ_2 are arbitrary representations of the algebra A , then

$$\mathcal{T}_{\varphi_1}(\zeta_1) \mathcal{T}_{\varphi_2}(\zeta_2) = \mathcal{T}_{\varphi_2}(\zeta_2) \mathcal{T}_{\varphi_1}(\zeta_1) \quad (3.22)$$

for all $\zeta_1, \zeta_2 \in \mathbb{C}^{\times}$.

Let a be an invertible group-like element of A which commutes with t . Using the equation

$$\mathcal{R}^{12}a^1a^2 = a^1a^2\mathcal{R}^{12},$$

we obtain

$$\mathcal{R}^{12}t^1a^1a^2 = a^1a^2\mathcal{R}^{12}t^1.$$

Rewriting this relation as

$$(a^1)^{-1}((\mathcal{R}^{12}t^1)a^2)a^1 = a^2(\mathcal{R}^{12}t^1), \quad (3.23)$$

and applying to both the sides the mapping $(\text{tr} \circ \varphi_\zeta) \otimes \text{id}$, we see that

$$\mathcal{T}_\varphi(\zeta)a = a\mathcal{T}_\varphi(\zeta).$$

for any invertible group-like element $a \in A$ commuting with the twist element t .

Assume that the vector space V is finite-dimensional of dimension ℓ , and $\{e_a\}$ is a basis of V . Denote by $F_\varphi(\zeta) \in \text{Mat}_\ell(\mathbb{C})$ the matrix of $\varphi_\zeta(t)$ with respect to the basis $\{e_a\}$. It is clear that

$$\mathcal{T}_\varphi(\zeta) = \text{tr}(\mathcal{M}_\varphi(\zeta)F_\varphi(\zeta)), \quad (3.24)$$

where the matrix $\mathcal{M}_\varphi(\zeta) \in \text{Mat}_\ell(A)$ is defined in section 3.2.1, and, using the canonical embedding of the field \mathbb{C} into A , we treat the matrix $F_\varphi(\zeta)$ as an element of $\text{Mat}_\ell(A)$. Applying to both sides of equation (3.19) the mapping $\varphi_{\zeta_1} \otimes \varphi_{\zeta_2}$, we obtain in terms of the corresponding matrices

$$\mathbf{R}_\varphi(\zeta_1|\zeta_2)(F_\varphi(\zeta_1) \otimes F_\varphi(\zeta_2)) = (F_\varphi(\zeta_1) \otimes F_\varphi(\zeta_2))\mathbf{R}_\varphi(\zeta_1|\zeta_2).$$

In terms of matrix entries this equation coincides with equation (2.4).

3.3.2. Transfer operator. Let ψ be a representation of the algebra A in the vector space U . We define the *transfer operator* $T_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n)$ by the relation

$$\begin{aligned} T_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) &= (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n})(\mathcal{T}_\varphi(\zeta)) \\ &= ((\text{tr}_V \circ \varphi_\zeta) \otimes (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n}))(\mathcal{R}(t \otimes 1)), \end{aligned}$$

where η_1, \dots, η_n are nonzero complex numbers, and the mapping ψ_η is defined in the same way as φ_ζ . Equation (3.21) immediately gives

$$T_{\varphi,\psi}(\zeta_1|\eta_1, \dots, \eta_n)T_{\varphi,\psi}(\zeta_2|\eta_1, \dots, \eta_n) = T_{\varphi,\psi}(\zeta_2|\eta_1, \dots, \eta_n)T_{\varphi,\psi}(\zeta_1|\eta_1, \dots, \eta_n) \quad (3.25)$$

for all $\zeta_1, \zeta_2 \in \mathbb{C}^\times$.

In the case when φ is a finite-dimensional representation, we see that

$$\begin{aligned} T_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) &\stackrel{(3.24)}{=} \text{tr}(\mathbf{M}_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n)F_\varphi(\zeta)) \\ &\stackrel{(3.15)}{=} \text{tr}((\mathbf{M}_{\varphi,\psi}(\zeta|\eta_1) \square \dots \square \mathbf{M}_{\varphi,\psi}(\zeta|\eta_n))F_\varphi(\zeta)). \end{aligned}$$

Here the matrix $\mathbf{M}_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n)$ is defined in section 3.2.2 and the matrix $F_\varphi(\zeta)$ in section 3.3.1. In the case where equation (3.3) is satisfied, from the above relation it follows, in particular, that

$$T_{\varphi,\psi}(\zeta\nu|\eta_1\nu, \dots, \eta_n\nu) = T_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n)$$

for any $\nu \in \mathbb{C}^\times$.

Assume now that ψ is a finite-dimensional representation, $\{e_i\}$ is a basis of U , and $\{E_{ij}\}$ the corresponding basis of $\text{End}(U)$. We can write

$$T_{\varphi,\psi}(\zeta|\eta_1, \dots, \eta_n) = \sum_{\substack{i_1, \dots, i_n \\ j_1, \dots, j_n}} T_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n) \otimes E_{i_1 j_1} \otimes \dots \otimes E_{i_n j_n}$$

for some $T_{i_1 \dots i_n | j_1 \dots j_n}(\zeta | \eta_1, \dots, \eta_n) \in \text{End}(V)$, and define the matrix

$$\mathbb{T}_{\varphi, \psi}(\zeta | \eta_1, \dots, \eta_n) = (T_{i_1 \dots i_n | j_1 \dots j_n}(\zeta | \eta_1, \dots, \eta_n)).$$

Now we have

$$\begin{aligned} \mathbb{T}_{\varphi, \psi}(\zeta | \eta_1, \dots, \eta_n) &= \text{tr}_V(\mathbb{M}_{\varphi, \psi}(\zeta | \eta_1, \dots, \eta_n) \varphi_\zeta(t)) \\ &\stackrel{(3.17)}{=} \text{tr}_V(\mathbb{M}_{\varphi, \psi}(\zeta | \eta_1) \boxtimes \dots \boxtimes \mathbb{M}_{\varphi, \psi}(\zeta | \eta_n) \varphi_\zeta(t)), \end{aligned} \quad (3.26)$$

where the matrix $\mathbb{M}_{\varphi, \psi}(\zeta | \eta_1, \dots, \eta_n)$ is defined in section 3.2.2, and the mapping tr_V is applied to the matrix entries. As follows from (3.25) the matrices $\mathbb{T}_{\varphi, \psi}(\zeta | \eta_1, \dots, \eta_n)$ for different values of the spectral parameter ζ commute,

$$[\mathbb{T}_{\varphi, \psi}(\zeta | \eta_1, \dots, \eta_n), \mathbb{T}_{\varphi, \psi}(\zeta' | \eta_1, \dots, \eta_n)] = 0,$$

see relation (2.2).

3.4. L-operators. To formulate and prove functional relations we additionally need Q-operators. We start with L-operators which play in the construction of Q-operators the same role as monodromy operators in the construction of transfer operators.

3.4.1. Universal L-operator. First of all, we assume that the universal R-matrix \mathcal{R} of the algebra A is an element of the tensor product $A_+ \otimes A_-$, where A_+ and A_- are proper subalgebras of A . In particular, it is so when A is the quantum group associated with an affine Lie algebra, see, for example, [32, 44, 51, 63]. Certainly, any representation of A can be restricted to representations of A_+ and A_- . However, this does not give new interesting objects. To construct L-operators one uses representations of A_+ which cannot be extended to representations of A . Let ρ be such a representation of A_+ in a vector space W . We define the *universal L-operator* by the equation

$$\mathcal{L}_\rho(\zeta) = (\rho_\zeta \otimes \text{id})(\mathcal{R}),$$

where the mapping ρ_ζ is defined by the relation similar to (3.1). It is clear that $\mathcal{L}_\rho(\zeta)$ is an element of $\text{End}(W) \otimes A_-$.

In spite of the fact that the definition of the universal L-operator is very similar to the definition of the universal monodromy matrix, we could not obtain for the universal L-matrix all relations satisfied by the universal monodromy matrix. This is due to the fact that, to obtain such relations, we should have a representation of the whole algebra A . Moreover, in all known interesting cases ρ is an infinite-dimensional representation, so we cannot introduce the corresponding matrices.

In fact, to come to functional relations, one should choose representations ρ , defining Q-operators, to be related to representations φ , used to define the monodromy operators and the corresponding transfer operators. Presently, we do not have full understanding of how to do it. It seems that representations ρ should be obtained from representations φ via some limiting procedure, see [11, 37] and the discussion in section 4.6.

3.4.2. L-operator. Let ψ be a representation of the subalgebra A_- in the vector space U . To come to objects satisfying functional relations one uses as ψ the restriction to A_- of the representation used to define the corresponding monodromy and transfer operators. The L-operator $L_{\rho, \psi}(\zeta | \eta_1, \dots, \eta_n)$ is defined as

$$\begin{aligned} L_{\rho, \psi}(\zeta | \eta_1, \dots, \eta_n) &= (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n})(\mathcal{L}_\rho(\zeta)) \\ &= (\rho_\zeta \otimes (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n}))(\mathcal{R}), \end{aligned}$$

where η_1, \dots, η_n are some nonzero complex numbers and the mapping ψ_η is defined by the relation similar to (3.1). It is evident that $L_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n)$ is an element of $\text{End}(W) \otimes \text{End}(U)^{\otimes n} \cong \text{End}(W) \otimes \text{End}(U^{\otimes n})$. As well as for the monodromy matrix, see (3.11), one can see that

$$L_{\rho,\psi}(\zeta|\eta_1 \dots, \eta_n) = L_{\rho,\psi}^{1,2}(\zeta|\eta_1) \dots L_{\rho,\psi}^{1,n+1}(\zeta|\eta_n). \quad (3.27)$$

If equation (3.3) is satisfied, in the same way as for the case of R -operators, see (3.5), we obtain

$$L_{\rho,\psi}(\zeta\nu|\eta\nu) = L_{\rho,\psi}(\zeta|\eta),$$

and, therefore,

$$L_{\rho,\psi}(\zeta|\eta) = L_{\rho,\psi}(\zeta\eta^{-1}),$$

where $L_{\rho,\psi}(\zeta) = L_{\rho,\psi}(\zeta|1)$. Equation (3.27) in this case gives

$$L_{\rho,\psi}(\zeta\nu|\eta_1\nu, \dots, \eta_n\nu) = L_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n).$$

Assume now that the representation ψ is finite-dimensional. Let $\{e_i\}$ be a basis of U and $\{E_{ij}\}$ the corresponding basis of $\text{End}(U)$. We can write

$$L_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) = \sum_{\substack{i_1, \dots, i_n \\ j_1, \dots, j_n}} L_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n) \otimes E_{i_1 j_1} \otimes \dots \otimes E_{i_n j_n},$$

where $L_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n)$ are elements of $\text{End}(W)$. Now we can introduce the matrix

$$\mathbb{L}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) = (L_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n)),$$

and be convinced that

$$\mathbb{L}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) = \mathbb{L}_{\rho,\psi}(\zeta|\eta_1) \boxtimes \dots \boxtimes \mathbb{L}_{\rho,\psi}(\zeta|\eta_n). \quad (3.28)$$

Detailed calculations giving the explicit forms of L -operators for the case of the quantum groups $U_q(\mathcal{L}(\mathfrak{sl}_2))$ and $U_q(\mathcal{L}(\mathfrak{sl}_3))$ can be found in the paper [19], and for the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_3, \mu))$, where again μ is the standard diagram automorphism of \mathfrak{sl}_3 of order 2, in the paper [20].

3.5. Q-operators.

3.5.1. *Universal Q-operator.* Given $\zeta \in \mathbb{C}^\times$, we define the *universal Q-operator* by the relation

$$\mathcal{Q}_\rho(\zeta) = \text{tr}_W(\mathcal{L}_\rho(\zeta)(\rho_\zeta(t) \otimes 1)).$$

One can easily see that

$$\mathcal{Q}_\rho(\zeta) = ((\text{tr}_W \circ \rho_\zeta) \otimes \text{id})(\mathcal{R}(t \otimes 1)) = ((\text{tr}_W \circ \rho_\zeta) \otimes \text{id})(\mathcal{R}^{12}t^1).$$

It is clear that $\mathcal{Q}_\rho(\zeta)$ is an element of the algebra A_- .

Applying the mapping $(\text{tr} \circ \rho_{\zeta_1}) \otimes (\text{tr} \circ \rho_{\zeta_2}) \otimes \text{id}$ to both sides of equation (3.20), we obtain

$$\mathcal{Q}_\rho(\zeta_1)\mathcal{T}_\varphi(\zeta_2) = \mathcal{T}_\varphi(\zeta_2)\mathcal{Q}_\rho(\zeta_1). \quad (3.29)$$

Here and below we assume that the same twist element is used to define both the universal Q-operator and the universal transfer matrix. Note also that, using equation (3.23), one can show that

$$\mathcal{Q}_\rho(\zeta)a = a\mathcal{Q}_\rho(\zeta)$$

for any invertible group-like element a which commutes with the twist element t . However, using only equation (3.20), one cannot prove the commutativity of $\mathcal{Q}_\rho(\zeta)$ for different values of the spectral parameter, because ρ cannot be extended to a representation of the whole algebra A . Here the following fact appears to be useful [58].

Let ρ_1 and ρ_2 be two representations of the algebra A_+ in vector spaces W_1 and W_2 , respectively, and $\rho_{1\zeta_1}$ and $\rho_{2\zeta_2}$ the mappings constructed by the relations similar to (3.1). We have

$$\begin{aligned} \mathcal{Q}_{\rho_1}(\zeta_1)\mathcal{Q}_{\rho_2}(\zeta_2) &= \left[((\text{tr}_{W_1} \circ \rho_{1\zeta_1}) \otimes \text{id})(\mathcal{R}^{13}t^1) \right] \left[((\text{tr}_{W_2} \circ \rho_{2\zeta_2}) \otimes \text{id})(\mathcal{R}^{23}t^2) \right] \\ &\stackrel{(E.4)}{=} \left((\text{tr}_{W_1 \otimes W_2} \circ (\rho_{1\zeta_1} \otimes \rho_{2\zeta_2})) \otimes \text{id} \right) \left(\mathcal{R}^{13}t^1 \mathcal{R}^{23}t^2 \right). \end{aligned}$$

Using equations (C.3) and (3.18), we obtain

$$\mathcal{R}^{13}t^1 \mathcal{R}^{23}t^2 = [(\Delta \otimes \text{id})(\mathcal{R})][(\Delta \otimes \text{id})(t \otimes 1)] = (\Delta \otimes \text{id})(\mathcal{R}(t \otimes 1)).$$

Hence, one can write

$$\mathcal{Q}_{\rho_1}(\zeta_1)\mathcal{Q}_{\rho_2}(\zeta_2) = \left((\text{tr}_{W_1 \otimes W_2} \circ (\rho_{1\zeta_1} \otimes \rho_{2\zeta_2})) \otimes \text{id} \right) \circ (\Delta \otimes \text{id})(\mathcal{R}(t \otimes 1)),$$

and, finally,

$$\mathcal{Q}_{\rho_1}(\zeta_1)\mathcal{Q}_{\rho_2}(\zeta_2) = \left((\text{tr}_{W_1 \otimes W_2} \circ (\rho_{1\zeta_1} \otimes_\Delta \rho_{2\zeta_2})) \otimes \text{id} \right) (\mathcal{R}(t \otimes 1)). \quad (3.30)$$

In a similar way one can obtain expressions for other products. For example,

$$\mathcal{T}_\varphi(\zeta_1)\mathcal{Q}_\rho(\zeta_2) = \left((\text{tr}_{V \otimes W} \circ (\varphi_{\zeta_1} \otimes_\Delta \rho_{\zeta_2})) \otimes \text{id} \right) (\mathcal{R}(t \otimes 1)). \quad (3.31)$$

3.5.2. *Q-operator.* We define the *Q-operator* $\mathcal{Q}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n)$ by the relation

$$\mathcal{Q}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) = \text{tr}_W(L_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n)).$$

It is evident that

$$\begin{aligned} \mathcal{Q}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) &= (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n})(\mathcal{Q}_\rho(\zeta)) \\ &= \left((\text{tr}_W \circ \rho_\zeta) \otimes (\psi_{\eta_1} \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \psi_{\eta_n}) \right) (\mathcal{R}(t \otimes 1)). \end{aligned}$$

Assume now that ψ is a finite-dimensional representation, $\{e_i\}$ is a basis of the representation space U , and $\{E_{ij}\}$ the corresponding basis of $\text{End}(U)$. We can write

$$Q_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) = \sum_{\substack{i_1, \dots, i_n \\ j_1, \dots, j_n}} Q_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n) \otimes E_{i_1 j_1} \otimes \dots \otimes E_{i_n j_n}$$

where $Q_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n)$ are the appropriate elements of $\text{End}(W)$, and define the matrix

$$\mathbb{Q}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) = (Q_{i_1 \dots i_n | j_1 \dots j_n}(\zeta|\eta_1, \dots, \eta_n)).$$

Now we have

$$\begin{aligned} \mathbb{Q}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) &= \text{tr}_W(\mathbb{L}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n) \rho_\zeta(t)) \\ &\stackrel{(3.28)}{=} \text{tr}_W(\mathbb{L}_{\rho,\psi}(\zeta|\eta_1) \boxtimes \dots \boxtimes \mathbb{L}_{\rho,\psi}(\zeta|\eta_n) \rho_\zeta(t)), \end{aligned} \quad (3.32)$$

where the matrix $\mathbb{L}_{\rho,\psi}(\zeta|\eta_1, \dots, \eta_n)$ is defined in section 3.4, and tr_W is applied to the matrix entries.

From equation (3.29) we obtain

$$[\mathbb{Q}_{\rho,\psi}(\zeta_1|\eta_1, \dots, \eta_n), T_{\varphi,\psi}(\zeta_2|\eta_1, \dots, \eta_n)] = 0,$$

or, in terms of the corresponding matrices,

$$[\mathbb{Q}_{\rho,\psi}(\zeta_1|\eta_1, \dots, \eta_n), \mathbb{T}_{\varphi,\psi}(\zeta_2|\eta_1, \dots, \eta_n)] = 0,$$

see relation (2.15).

Further progress in obtaining functional relations can be achieved only by using the properties of the specific representations of concrete quasitriangular Hopf algebras. The corresponding calculations were given for the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ in the papers [15, 16, 26], for the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_3))$ [11], for the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_{2|1}))$ in the paper [17], see also [1, 47–49, 58]. In the next section we reconsider the case of $U_q(\mathcal{L}(\mathfrak{sl}_2))$, having in mind to fill certain gaps of [1, 15, 16, 26] and to derive the full set of functional relations in the model-independent form.

4. EXAMPLE RELATED TO THE SIX-VERTEX MODEL

As an example we consider the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$. To obtain objects related to integrable systems, we need representations of this quasitriangular Hopf algebra. The standard method here is to use the Jimbo's homomorphism [40] from $U_q(\mathcal{L}(\mathfrak{sl}_2))$ to the quantum group $U_q(\mathfrak{sl}_2)$, and then construct representations of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ from representations of $U_q(\mathfrak{sl}_2)$.

Depending on the sense of q , there are at least three definitions of a quantum group. According to the first definition, $q = \exp \hbar$, where \hbar is an indeterminate, according to the second one, q is indeterminate, and according to the third one, $q = \exp \hbar$, where \hbar is a complex number such that $q \neq 0, \pm 1$. In the first case a quantum group is a $\mathbb{C}[[\hbar]]$ -algebra, in the second case a $\mathbb{C}(q)$ -algebra, and in the third case it is just a complex algebra. It seems that to define traces appropriately, it is convenient to use the third definition. Therefore, we define the quantum group as a \mathbb{C} -algebra, see, for example, the books [36, 41].

4.1. Quantum group $U_q(\mathfrak{sl}_2)$.

4.1.1. *Definition.* Let \hbar be a complex number such that $q = \exp \hbar \neq 0, \pm 1$. We assume that $q^\nu, \nu \in \mathbb{C}$, means the complex number $\exp(\hbar\nu)$. The quantum group $U_q(\mathfrak{sl}_2)$ is a \mathbb{C} -algebra generated by the elements E, F , and $q^{\nu H}, \nu \in \mathbb{C}$, with the following defining relations:

$$\begin{aligned} q^0 &= 1, & q^{\nu_1 H} q^{\nu_2 H} &= q^{(\nu_1 + \nu_2)H}, \\ q^{\nu H} E q^{-\nu H} &= q^{2\nu} E, & q^{\nu H} F q^{-\nu H} &= q^{-2\nu} F, \\ [E, F] &= \kappa_q^{-1} (q^H - q^{-H}). \end{aligned}$$

Here and below $\kappa_q = q - q^{-1}$. Note that $q^{\nu H}$ is just a notation, there is no an element $H \in U_q(\mathfrak{sl}_2)$. In fact, it is constructive to identify H with the standard Cartan element of the Lie algebra \mathfrak{sl}_2 , and νH with a general element of the Cartan subalgebra $\mathfrak{h} = \mathbb{C}H$. Using such interpretation, one can say that $q^{\nu H}$ is a set of generators parameterized by the elements of the Cartan subalgebra \mathfrak{h} .

The quantum group $U_q(\mathfrak{sl}_2)$ is also a Hopf algebra with the comultiplication

$$\begin{aligned} \Delta(q^{\nu H}) &= q^{\nu H} \otimes q^{\nu H}, \\ \Delta(E) &= E \otimes 1 + q^{-H} \otimes E, & \Delta(F) &= F \otimes q^H + 1 \otimes F, \end{aligned}$$

and the correspondingly defined counit and antipode.⁷

The monomials $E^r F^s q^{\nu H}$ for $r, s \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$ form a basis of $U_q(\mathfrak{sl}_2)$. There is one more basis defined with the help of the *quantum Casimir element* C which has the form⁸

$$C = q^{H-1} + q^{-H+1} + \kappa_q^2 EF = q^{H+1} + q^{-H-1} + \kappa_q^2 FE.$$

Here and below we use the notation $q^{\nu H + \mu} = q^\mu q^{\nu H}, \nu, \mu \in \mathbb{C}$. One can verify that C belongs to the center of $U_q(\mathfrak{sl}_2)$. It is clear that the monomials $E^{r+1} C^s q^{\nu H}, F^{r+1} C^s q^{\nu H}$ and $C^s q^{\nu H}$ for $r, s \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$ also form a basis of $U_q(\mathfrak{sl}_2)$. This basis is convenient to define traces on $U_q(\mathfrak{sl}_2)$.

4.1.2. *Verma representation.* Given $\mu \in \mathbb{C}$, let \tilde{V}^μ be a free vector space generated by the set $\{v_0, v_1, \dots\}$. Introduce the notation

$$[v]_q = \frac{q^v - q^{-v}}{q - q^{-1}}, \quad v \in \mathbb{C}.$$

One can show that the relations

$$q^{\nu H} v_n = q^{\nu(\mu - 2n)} v_n, \quad E v_n = [n]_q [\mu - n + 1]_q v_{n-1}, \quad F v_n = v_{n+1} \quad (4.1)$$

endow \tilde{V}^μ with the structure of a left $U_q(\mathfrak{sl}_2)$ -module. The module \tilde{V}^μ is isomorphic to the Verma module with the highest weight whose action on H gives μ .

We denote the representation of $U_q(\mathfrak{sl}_2)$ corresponding to the module \tilde{V}^μ by $\tilde{\pi}^\mu$. If μ equals a non-negative integer m , the linear hull of the vectors v_n with $n > m$ is a submodule of \tilde{V}^m isomorphic to the module \tilde{V}^{-m-2} . We denote the corresponding finite-dimensional quotient module by V^m and the corresponding representation by π^m .

⁷There are a few different equivalent choices for comultiplication, counit and antipode in $U_q(\mathfrak{sl}_2)$. Since we are going to use the Khoroshkin–Tolstoy expression for the universal R -matrix, we follow the convention of the paper [63].

⁸We use a nonstandard, but convenient for our purposes, normalization of C .

It is easy to see that for the quantum Casimir element we have

$$\tilde{\pi}^\mu(C) = q^{\mu+1} + q^{-\mu-1}, \quad \pi^m(C) = q^{m+1} + q^{-m-1} \quad (4.2)$$

for any $\mu \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$.

4.2. Quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$.

4.2.1. *Definition.* We start with the quantum group $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$. The reason is that the expression for the universal R -matrix given by Khoroshkin and Tolstoy [63] is valid for the case of $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$.

First, let us describe the root system of $\widehat{\mathcal{L}}(\mathfrak{sl}_2)$. The Cartan subalgebra of $\widehat{\mathcal{L}}(\mathfrak{sl}_2)$ is

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathfrak{h} = \mathbb{C}H$ is the standard Cartan subalgebra of \mathfrak{sl}_2 , c is the central element, and d is the derivation [43]. Define the Cartan elements

$$h_0 = c - H, \quad h_1 = H, \quad (4.3)$$

so that one has

$$\widehat{\mathfrak{h}} = \mathbb{C}h_0 \oplus \mathbb{C}h_1 \oplus \mathbb{C}d.$$

The simple positive roots $\alpha_0, \alpha_1 \in \widehat{\mathfrak{h}}^*$ are given by the equations

$$\begin{aligned} \alpha_j(h_i) &= a_{ij}, \\ \alpha_0(d) &= 1, \quad \alpha_1(d) = 0, \end{aligned}$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

is the Cartan matrix of the Lie algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_2)$. The full root system Δ of $\widehat{\mathcal{L}}(\mathfrak{sl}_2)$ is the disjoint union of the system of positive roots

$$\begin{aligned} \Delta_+ &= \{\alpha_0 + k(\alpha_0 + \alpha_1) \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{\alpha_1 + k(\alpha_0 + \alpha_1) \mid k \in \mathbb{Z}_{\geq 0}\} \\ &\quad \cup \{k(\alpha_0 + \alpha_1) \mid k \in \mathbb{Z}_{>0}\} \end{aligned}$$

and the system of negative roots $\Delta_- = -\Delta_+$ [43].

Let again \hbar be a complex number, such that $q = \exp \hbar \neq 0, \pm 1$. The quantum group $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ is a \mathbb{C} -algebra generated by the elements $e_i, f_i, i = 0, 1$, and $q^x, x \in \widehat{\mathfrak{h}}$, with the relations

$$q^0 = 1, \quad q^{x_1} q^{x_2} = q^{x_1+x_2}, \quad (4.4)$$

$$q^x e_i q^{-x} = q^{\alpha_i(x)} e_i, \quad q^x f_i q^{-x} = q^{-\alpha_i(x)} f_i, \quad (4.5)$$

$$[e_i, f_j] = \kappa_q^{-1} \delta_{ij} (q^{h_i} - q^{-h_i}) \quad (4.6)$$

satisfied for all i and j , and the Serre relations

$$e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0, \quad (4.7)$$

$$f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (4.8)$$

satisfied for all distinct i and j .

The quantum group $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ is a Hopf algebra with the comultiplication Δ defined by the relations

$$\Delta(q^x) = q^x \otimes q^x, \quad (4.9)$$

$$\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i, \quad (4.10)$$

and with the correspondingly defined counit and antipode.

To give the definition of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$, we first introduce the Hopf subalgebra $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_2))$ of $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ generated by $e_i, f_i, i = 0, 1$, and $q^x, x \in \tilde{\mathfrak{h}}$, where

$$\tilde{\mathfrak{h}} = \mathbb{C}h_0 \oplus \mathbb{C}h_1 = \mathbb{C}H \oplus \mathbb{C}c.$$

The quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ can be defined as the quotient algebra of $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_2))$ by the two-sided ideal generated by the elements of the form $q^{\nu c} - 1, \nu \in \mathbb{C}^\times$. In terms of generators and relations the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ is a \mathbb{C} -algebra generated by the elements $e_i, f_i, i = 0, 1$, and $q^x, x \in \tilde{\mathfrak{h}}$, with relations (4.4)–(4.8) and

$$q^{\nu(h_0+h_1)} = q^{\nu c} = 1, \quad (4.11)$$

where $\nu \in \mathbb{C}^\times$. It is a Hopf algebra with the comultiplication defined by (4.9), (4.10) and with the correspondingly defined counit and antipode. One of the reasons to use the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ instead of $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ is that $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ has no finite-dimensional representations with a nontrivial action of $q^{\nu(h_0+h_1)} = q^{\nu c}$.

4.2.2. *Useful basis.* We call a nonzero element $a \in U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ a *root element* corresponding to the root $\gamma \in \widehat{\mathfrak{h}}^*$ if

$$q^x a q^{-x} = q^{\gamma(x)} a$$

for any $x \in \widehat{\mathfrak{h}}$. It can be shown that for any $\gamma \in \Delta$ there is a nonzero root element, and this element is unique up to multiplication by a nonzero scalar factor. It is clear that the generators e_i and f_i correspond to the roots α_i and $-\alpha_i$ respectively. Choose for each root of Δ a root element, and denote the root element corresponding to a positive root γ by e_γ and the root element corresponding to a negative root $-\gamma$ by f_γ . Assume that some total order \prec of positive roots is fixed. It appears that the monomials of the form

$$e_{\gamma_1}^{k_1} \dots e_{\gamma_r}^{k_r} f_{\delta_1}^{\ell_1} \dots f_{\delta_s}^{\ell_s} q^x,$$

where $\gamma_1 \prec \dots \prec \gamma_r$ and $\delta_1 \prec \dots \prec \delta_s$, form a basis of $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$.

Let us describe the method to construct the root elements corresponding to the roots of Δ used by Khoroshkin and Tolstoy [63]. It can be shown [45] that the appearing root elements are closely related to the quantum group generators introduced by Drinfeld [35].

It is customary to denote $\alpha = \alpha_1$ and $\delta = \alpha_0 + \alpha_1$, so that the simple positive roots are now $\delta - \alpha$ and α . Then the system of positive roots is

$$\Delta_+ = \{\alpha + k\delta \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{k\delta \mid k \in \mathbb{Z}_{> 0}\} \cup \{\delta - \alpha + k\delta \mid k \in \mathbb{Z}_{\geq 0}\}.$$

For the simple roots we choose

$$e_\alpha = e_{\alpha_1}, \quad e_{\delta-\alpha} = e_{\alpha_0}, \quad f_\alpha = f_{\alpha_1}, \quad f_{\delta-\alpha} = f_{\alpha_0}. \quad (4.12)$$

Now we define the root element corresponding to the root δ putting

$$e'_\delta = e_\alpha e_{\delta-\alpha} - q^{-2} e_{\delta-\alpha} e_\alpha. \quad (4.13)$$

Here we use the prime because to construct the universal R -matrix we redefine the root elements corresponding to the roots $k\delta$ and $-k\delta$ and denote by $e_{k\delta}$ and $f_{k\delta}$ the result of the redefinition. The remaining root elements corresponding to the positive roots are defined recursively by the relations

$$e_{\alpha+k\delta} = [2]_q^{-1} (e_{\alpha+(k-1)\delta} e'_\delta - e'_\delta e_{\alpha+(k-1)\delta}), \quad (4.14)$$

$$e_{(\delta-\alpha)+k\delta} = [2]_q^{-1} (e'_\delta e_{(\delta-\alpha)+(k-1)\delta} - e_{(\delta-\alpha)+(k-1)\delta} e'_\delta), \quad (4.15)$$

$$e'_{k\delta} = e_{\alpha+(k-1)\delta} e_{\delta-\alpha} - q^{-2} e_{\delta-\alpha} e_{\alpha+(k-1)\delta}. \quad (4.16)$$

The root elements corresponding to the negative roots are defined with the help of the relations

$$f'_\delta = f_{\delta-\alpha} f_\alpha - q^2 f_\alpha f_{\delta-\alpha}, \quad (4.17)$$

$$f_{\alpha+k\delta} = [2]_q^{-1} (f'_\delta f_{\alpha+(k-1)\delta} - f_{\alpha+(k-1)\delta} f'_\delta), \quad (4.18)$$

$$f_{(\delta-\alpha)+k\delta} = [2]_q^{-1} (f_{(\delta-\alpha)+(k-1)\delta} f'_\delta - f'_\delta f_{(\delta-\alpha)+(k-1)\delta}), \quad (4.19)$$

$$f'_{k\delta} = f_{\delta-\alpha} f_{\alpha+(k-1)\delta} - q^2 f_{\alpha+(k-1)\delta} f_{\delta-\alpha}. \quad (4.20)$$

The coefficients in (4.14), (4.15), (4.18) and (4.19) are chosen in such a way that for $\gamma = \alpha$ and $\gamma = \delta - \alpha$ we have

$$[e_{\gamma+k\delta}, f_{\gamma+k\delta}] = \kappa_q^{-1} (q^{h_\gamma} - q^{-h_\gamma}),$$

where $h_\alpha = h_1$ and $h_{\delta-\alpha} = h_0$.

The root elements $e_{k\delta}$ needed for the construction of the universal R -matrix are related to the root elements $e'_{k\delta}$ by the equation

$$\kappa_q e_\delta(x) = \log(1 + \kappa_q e'_\delta(x)), \quad (4.21)$$

where

$$e'_\delta(x) = \sum_{k=1}^{\infty} e'_{k\delta} x^{-k}, \quad e_\delta(x) = \sum_{k=1}^{\infty} e_{k\delta} x^{-k}.$$

The root elements $f_{k\delta}$ are defined with the help of the equation

$$-\kappa_q f_\delta(x) = \log(1 - \kappa_q f'_\delta(x)), \quad (4.22)$$

where

$$f'_\delta(x) = \sum_{k=1}^{\infty} f'_{k\delta} x^{-k}, \quad f_\delta(x) = \sum_{k=1}^{\infty} f_{k\delta} x^{-k}.$$

4.2.3. Universal R -matrix. We follow here the approach developed by Khoroshkin and Tolstoy [63]. Although this is not clearly stated in the paper [63], Khoroshkin and Tolstoy define a quantum group as a $\mathbb{C}[[\hbar]]$ -algebra. In fact, one can use the expression for the universal R -matrix from the paper [63] also for the case of a quantum group defined as a \mathbb{C} -algebra having in mind that in this case a quantum group is quasitriangular only in some restricted sense. Namely, all the relations involving the universal R -matrix should be considered as valid only for the weight $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ -modules, see in this respect the paper [62] and the discussion below. Remind that a $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ -module V is a *weight module* if

$$V = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^*} V_\lambda,$$

where

$$V_\lambda = \{v \in V \mid q^x v = q^{\lambda(x)} v \text{ for any } x \in \widehat{\mathfrak{h}}\}.$$

The same terminology is used for the corresponding representations.

According to the paper [63], one starts with choosing some normal order \prec of the positive roots of $\widehat{\mathcal{L}}(\mathfrak{sl}_2)$. In general, one says that a system of positive roots is supplied with a normal order if its roots are totally ordered in such a way that

- (i) all multiple roots follow each other in an arbitrary order;
- (ii) each non-simple root $\alpha + \beta$, where α is not proportional to β , is placed between α and β .

In our case a normal order is fixed uniquely if we require

$$\alpha + k\delta \prec \ell\delta \prec (\delta - \alpha) + m\delta,$$

and in accordance with it the roots go as

$$\alpha, \alpha + \delta, \dots, \alpha + k\delta, \dots, \delta, 2\delta, \dots, \ell\delta, \dots, (\delta - \alpha) + m\delta, \dots, (\delta - \alpha) + \delta, \delta - \alpha.$$

The expression for the universal R -matrix, obtained by Khoroshkin and Tolstoy, has the form

$$\mathcal{R} = \mathcal{R}_{\prec\delta} \mathcal{R}_{\sim\delta} \mathcal{R}_{\succ\delta} \mathcal{K}. \quad (4.23)$$

The first factor is the product over $k \in \mathbb{Z}_{\geq 0}$ of the q -exponentials

$$\mathcal{R}_{\alpha,k} = \exp_{q^{-2}} (\kappa_q e_{\alpha+k\delta} \otimes f_{\alpha+k\delta})$$

in the order coinciding with the chosen normal order of the roots $\alpha + k\delta$. Here the q -exponential is defined as

$$\exp_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n)_q},$$

where

$$(n)_q! = (1)_q(2)_q \dots (n)_q, \quad (n)_q = \frac{q^n - 1}{q - 1}.$$

The factor $\mathcal{R}_{\sim\delta}$ is given by the expression

$$\mathcal{R}_{\sim\delta} = \exp \left(\kappa_q \sum_{k=1}^{\infty} \frac{k}{[2k]_q} e_{k\delta} \otimes f_{k\delta} \right). \quad (4.24)$$

The factor $\mathcal{R}_{\succ\delta}$ is the product over $k \in \mathbb{Z}_{\geq 0}$ of the q -exponentials

$$\mathcal{R}_{\delta-\alpha,k} = \exp_{q^{-2}} (\kappa_q e_{(\delta-\alpha)+k\delta} \otimes f_{(\delta-\alpha)+k\delta})$$

in the order coinciding with the chosen normal order of the roots $(\delta - \alpha) + k\delta$.

The last factor \mathcal{K} is not defined as an element of $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2)) \otimes U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$. However, one can define its action on the tensor product of any two weight $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ -modules. Let V and U be weight $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$ -modules with the weight decompositions

$$V = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^*} V_\lambda, \quad U = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} U_\mu.$$

The action of \mathcal{K} on $V \otimes U$ is defined by the relation

$$\mathcal{K} v \otimes u = q^{\lambda(h_\alpha)\mu(h_\alpha)/2 + \lambda(c)\mu(d) + \lambda(d)\mu(c)} v \otimes u, \quad (4.25)$$

where $v \in V_\lambda$ and $u \in U_\mu$. Slightly abusing the notation, we denote the corresponding operator by $(\varphi \otimes \psi)(\mathcal{K})$, where φ and ψ are the representations corresponding to the modules V and U respectively. If the module U is finite-dimensional, $\{e_r\}$ is a basis of U formed by weight vectors, and $\{E_{rs}\}$ is the corresponding basis of $\text{End}(U)$, we have

$$(\varphi \otimes \psi)(\mathcal{K}) = \sum_r \varphi(q^{\mu_r(h_\alpha)h_\alpha/2 + \mu_r(c)d + \mu_r(d)c}) \otimes E_{rr}, \quad (4.26)$$

where μ_r is the weight of e_r .

Note that in the case when we define a quantum group as a $\mathbb{C}[[\hbar]]$ -algebra, \mathcal{K} is an element of its tensor square of the form

$$\mathcal{K} = q^{h_\alpha \otimes h_\alpha / 2 + c \otimes d + d \otimes c},$$

and the notation $(\varphi \otimes \psi)(\mathcal{K})$ has a straightforward sense.

In the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ we again define the elements e_γ and f_γ , $\gamma \in \Delta_+$, by relations (4.12)–(4.22). The universal R -matrix is again defined by equation (4.23), where the factors $\mathcal{R}_{\prec\delta}$, $\mathcal{R}_{\sim\delta}$ and $\mathcal{R}_{\succ\delta}$ are defined in the same way as in the case of the quantum group $U_q(\widehat{\mathcal{L}}(\mathfrak{sl}_2))$, while for the factor \mathcal{K} we have

$$\mathcal{K} v \otimes u = q^{\lambda(h_\alpha)\mu(h_\alpha)/2} v \otimes u \quad (4.27)$$

instead of equation (4.25), and

$$(\varphi \otimes \psi)(\mathcal{K}) = \sum_r \varphi(q^{\mu_r(h_\alpha)h_\alpha/2}) \otimes E_{rr} \quad (4.28)$$

instead of equation (4.26).

4.3. R -operators. First of all, we assume that actual spectral parameters are complex numbers u and v_i such that

$$\zeta = q^u = e^{\hbar u}, \quad \eta_i = q^{v_i} = e^{\hbar v_i}. \quad (4.29)$$

This convention allows us to uniquely define arbitrary complex powers of ζ and η_i .

To construct R -operators we need representations of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$. We start with the Jimbo's homomorphism [40]

$$\varphi : U_q(\mathcal{L}(\mathfrak{sl}_2)) \rightarrow U_q(\mathfrak{sl}_2)$$

defined by the equations

$$\varphi(q^{vh_\alpha}) = q^{vH}, \quad \varphi(e_\alpha) = E, \quad \varphi(f_\alpha) = F, \quad (4.30)$$

$$\varphi(q^{vh_{\delta-\alpha}}) = q^{-vH}, \quad \varphi(e_{\delta-\alpha}) = F, \quad \varphi(f_{\delta-\alpha}) = E. \quad (4.31)$$

Let $\tilde{\pi}^\mu$ be the highest weight infinite-dimensional representation of $U_q(\mathfrak{sl}_2)$ with the highest weight μ described above. We define a representation $\tilde{\varphi}^\mu$ of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ as

$$\tilde{\varphi}^\mu = \tilde{\pi}^\mu \circ \varphi.$$

One more necessary ingredient is a \mathbb{Z} -gradation of $U_q(\mathcal{L}(\mathfrak{sl}_2))$. We define it assuming that the generators q^x belong to the zero grade subspace, the generators e_i belong to the graded subspaces with the grading indices s_i , and the generators f_i belong to the graded subspaces with the grading indices $-s_i$. For the mapping Φ_v , defined by (D.1), we have

$$\Phi_v(q^x) = q^x, \quad \Phi_v(e_i) = v^{s_i} e_i, \quad \Phi_v(f_i) = v^{-s_i} f_i. \quad (4.32)$$

Note that with this definition of a \mathbb{Z} -gradation of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ the universal R -matrix (4.23) satisfies equation (3.3). Below we denote $s = s_0 + s_1$. Now, given $\zeta \in \mathbb{C}^\times$, we define the representation $\tilde{\varphi}_\zeta^\mu$ as

$$\tilde{\varphi}_\zeta^\mu = \tilde{\pi}^\mu \circ \varphi \circ \Phi_\zeta.$$

Slightly abusing the notation, we denote the corresponding $U_q(\mathcal{L}(\mathfrak{sl}_2))$ -modules by \tilde{V}^μ and \tilde{V}_ζ^μ . Taking into account (4.1), (4.30), (4.31), and (4.32), we see that for the module

\tilde{V}_ζ^μ one has

$$q^{vh_\alpha} v_n = q^{v(\mu-2n)} v_n, \quad q^{vh_{\delta-\alpha}} v_n = q^{-v(\mu-2n)} v_n, \quad (4.33)$$

$$e_\alpha v_n = \zeta^{s_1} [n]_q [\mu - n + 1]_q v_{n-1}, \quad e_{\delta-\alpha} v_n = \zeta^{s_0} v_{n+1}, \quad (4.34)$$

$$f_\alpha v_n = \zeta^{-s_1} v_{n+1}, \quad f_{\delta-\alpha} v_n = \zeta^{-s_0} [n]_q [\mu - n + 1]_q v_{n-1}. \quad (4.35)$$

In the case when μ equals a non-negative integer m we denote the corresponding finite-dimensional representations by φ^m and φ_ζ^m , and the modules by V^m and V_ζ^m .

Now we denote

$$R_m(\zeta_{12}) = (\varphi_{\zeta_1}^m \otimes \varphi_{\zeta_2}^m)(\mathcal{R})$$

and shortly describe how to find an explicit expression for $R(\zeta) = R_1(\zeta)$. We refer the reader to the paper [19] for more details.

The representation π^1 of $U_q(\mathfrak{sl}_2)$ is two-dimensional and we have

$$\pi^1(q^{vH}) = q^v E_{11} + q^{-v} E_{22}, \quad \pi^1(E) = E_{12}, \quad \pi^1(F) = E_{21}. \quad (4.36)$$

Here and below E_{ab} are elements of the basis of $\text{End}(\mathbb{C}^2)$ corresponding to the standard basis $\{e_a\}$ of \mathbb{C}^2 . Using (4.30), (4.31) and (4.32), we come to the relations

$$\varphi_\zeta^1(q^{vh_\alpha}) = q^v E_{11} + q^{-v} E_{22}, \quad \varphi_\zeta^1(q^{vh_{\delta-\alpha}}) = q^{-v} E_{11} + q^v E_{22}, \quad (4.37)$$

$$\varphi_\zeta^1(e_\alpha) = \zeta^{s_1} E_{12}, \quad \varphi_\zeta^1(e_{\delta-\alpha}) = \zeta^{s-s_1} E_{21}, \quad (4.38)$$

$$\varphi_\zeta^1(f_\alpha) = \zeta^{-s_1} E_{21}, \quad \varphi_\zeta^1(f_{\delta-\alpha}) = \zeta^{-s+s_1} E_{12}. \quad (4.39)$$

It follows from (4.13) and (4.17) that

$$\varphi_\zeta^1(e'_\delta) = \zeta^s (E_{11} - q^{-2} E_{22}), \quad \varphi_\zeta^1(f'_\delta) = \zeta^{-s} (E_{11} - q^2 E_{22}),$$

and the recursive definitions (4.14), (4.15), (4.18) and (4.19) give

$$\varphi_\zeta^1(e_{\alpha+k\delta}) = (-1)^k q^{-k} \zeta^{s_1+ks} E_{12}, \quad \varphi_\zeta^1(e_{(\delta-\alpha)+k\delta}) = (-1)^k q^{-k} \zeta^{(s-s_1)+ks} E_{21}, \quad (4.40)$$

$$\varphi_\zeta^1(f_{\alpha+k\delta}) = (-1)^k q^k \zeta^{-s_1-ks} E_{21}, \quad \varphi_\zeta^1(f_{(\delta-\alpha)+k\delta}) = (-1)^k q^k \zeta^{-(s-s_1)-ks} E_{12}. \quad (4.41)$$

Starting from (4.16) and (4.40), we come to the equation

$$\varphi_\zeta^1(e'_{k\delta}) = (-1)^{k-1} q^{-k+1} \zeta^{ks} (E_{11} - q^{-2} E_{22}).$$

Taking into account (4.21), we obtain

$$\varphi_\zeta^1(e_{k\delta}) = (-1)^{k-1} [k]_q \frac{\zeta^{ks}}{k} (E_{11} - q^{-2k} E_{22}). \quad (4.42)$$

In a similar way, starting from (4.20) and (4.41) and taking into account (4.22), we determine that

$$\varphi_\zeta^1(f_{k\delta}) = (-1)^{k-1} [k]_q \frac{\zeta^{-ks}}{k} (E_{11} - q^{2k} E_{22}). \quad (4.43)$$

Now we can obtain expressions for the images of the factors entering the Khoroshkin–Tolstoy formula (4.23) for the universal R -matrix. To find expressions for the images of the factors $\mathcal{R}_{\prec\delta}$ and $\mathcal{R}_{\succ\delta}$, we use the identities

$$(E_{12})^n = 0, \quad (E_{21})^n = 0 \quad (4.44)$$

valid for any integer $n > 1$. Using (4.40) and (4.41) and summing up the arising geometric series, we find

$$(\varphi_{\zeta_1}^1 \otimes \varphi_{\zeta_2}^1)(\mathcal{R}_{\prec\delta}) = 1 \otimes 1 + \kappa_q \frac{\zeta_2^{s_1}}{1 - \zeta_2^s} E_{12} \otimes E_{21}, \quad (4.45)$$

$$(\varphi_{\zeta_1}^1 \otimes \varphi_{\zeta_2}^1)(\mathcal{R}_{\succ\delta}) = 1 \otimes 1 + \kappa_q \frac{\zeta_{12}^{s-s_1}}{1 - \zeta_{12}^s} E_{21} \otimes E_{12}. \quad (4.46)$$

Using (4.42) and (4.43), we obtain for the image of the factor $\mathcal{R}_{\sim\delta}$ the expression

$$\begin{aligned} (\varphi_{\zeta_1}^1 \otimes \varphi_{\zeta_2}^1)(\mathcal{R}_{\sim\delta}) &= e^{\lambda_2(q\zeta_{12}^s) - \lambda_2(q^{-1}\zeta_{12}^s)} \left[E_{11} \otimes E_{11} \right. \\ &\quad \left. + \frac{1 - q^2\zeta_{12}^s}{1 - \zeta_{12}^s} E_{11} \otimes E_{22} + \frac{1 - \zeta_{12}^s}{1 - q^{-2}\zeta_{12}^s} E_{22} \otimes E_{11} + E_{22} \otimes E_{22} \right], \end{aligned} \quad (4.47)$$

where

$$\lambda_2(\zeta) = \sum_{m \in \mathbb{Z}_+} \frac{1}{q^k + q^{-k}} \frac{\zeta^k}{k} = \sum_{k=1}^{\infty} \frac{1}{[2]_{q^k}} \frac{\zeta^k}{k}. \quad (4.48)$$

The simplest part of the calculations is to obtain an expression for the action of the factor \mathcal{K} on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$. As before, slightly abusing the notation, we denote the corresponding operator by $\varphi_{\zeta_1}^1 \otimes \varphi_{\zeta_2}^1(\mathcal{K})$. Let μ_1 and μ_2 be the weights of the vectors e_1 and e_2 forming the standard basis of \mathbb{C}^2 . It follows from (4.37) that

$$\mu_1(h_\alpha) = 1, \quad \mu_2(h_\alpha) = -1, \quad (4.49)$$

and equation (4.28) gives

$$\begin{aligned} (\varphi_{\zeta_1}^1 \otimes \varphi_{\zeta_2}^1)(\mathcal{K}) &= q^{1/2} E_{11} \otimes E_{11} + q^{-1/2} E_{11} \otimes E_{22} \\ &\quad + q^{-1/2} E_{22} \otimes E_{11} + q^{1/2} E_{22} \otimes E_{22}. \end{aligned} \quad (4.50)$$

Now we have the expressions for all factors necessary to obtain the expression for $R(\zeta) = R_1(\zeta)$. After simple calculations we determine that

$$\begin{aligned} R(\zeta) &= q^{1/2} e^{\lambda_2(q\zeta^s) - \lambda_2(q^{-1}\zeta^s)} \left[E_{11} \otimes E_{11} + E_{22} \otimes E_{22} \right. \\ &\quad \left. + \frac{q^{-1}(1 - \zeta^s)}{1 - q^{-2}\zeta^s} (E_{11} \otimes E_{22} + E_{22} \otimes E_{11}) \right. \\ &\quad \left. + \frac{1 - q^{-2}}{1 - q^{-2}\zeta^s} (\zeta^{s_1} E_{12} \otimes E_{21} + \zeta^{s_0} E_{21} \otimes E_{12}) \right]. \end{aligned} \quad (4.51)$$

It is instructive, using the identity

$$\lambda_2(q\zeta) + \lambda_2(q^{-1}\zeta) = -\log(1 - \zeta), \quad (4.52)$$

to rewrite the expression for $R(\zeta)$ as

$$\begin{aligned} R(\zeta) &= q^{-1/2} \zeta^{s/2} e^{\lambda_2(q\zeta^s) + \lambda_2(q^{-3}\zeta^s)} \left[(q\zeta^{-s/2} - q^{-1}\zeta^{s/2})(E_{11} \otimes E_{11} + E_{22} \otimes E_{22}) \right. \\ &\quad \left. + (\zeta^{-s/2} - \zeta^{s/2})(E_{11} \otimes E_{22} + E_{22} \otimes E_{11}) \right. \\ &\quad \left. + \kappa_q (\zeta^{-(s_0-s_1)/2} E_{12} \otimes E_{21} + \zeta^{(s_0-s_1)/2} E_{21} \otimes E_{12}) \right]. \end{aligned} \quad (4.53)$$

Note that we come to the most frequently used symmetric R -operator putting $s_0 = -1$ and $s_1 = -1$ and omitting the factor before the square bracket, compare with relations (2.8), (2.6) and (2.7).

4.4. Monodromy operators. Now we find an expression for the monodromy operator

$$M_{\varphi,\psi}(\zeta) = M_{\varphi,\psi}(\zeta|1) = (\varphi_\zeta \otimes \psi)(\mathcal{R})$$

choosing as φ the Jimbo's homomorphism and as ψ the representation φ^1 . In fact, we extend the notion of the monodromy operator allowing for using general homomorphisms instead of representations. To simplify notation, we write instead of $M_{\varphi,\varphi^1}(\zeta)$ just $M(\zeta)$.

Using relations (4.13), (4.30) and (4.31), we obtain

$$\varphi_\zeta(e'_\delta) = \kappa_q^{-1} q^{-1} \left[C - (q + q^{-1}) q^{-H} \right] \zeta^s.$$

Now, using definition (4.14), we come to the expression

$$\varphi_\zeta(e_{\alpha+k\delta}) = (-1)^k q^{-kH} E \zeta^{s_1+ks}, \quad (4.54)$$

while definition (4.15) gives

$$\varphi_\zeta(e_{(\delta-\alpha)+k\delta}) = (-1)^k F q^{-kH} \zeta^{(s-s_1)+ks}. \quad (4.55)$$

Relation (4.16) together with (4.31) and (4.54) lead to the equation

$$\begin{aligned} \varphi_\zeta(e'_{k\delta}) = \kappa_q^{-2} (-1)^{k-1} q^{-k} & \left[(q^k - q^{-k}) C q^{-(k-1)H} \right. \\ & \left. - (q^{k-1} - q^{-k+1}) q^{-(k-2)H} - (q^{k+1} - q^{-k-1}) q^{-kH} \right] \zeta^{ks}, \end{aligned}$$

and we have

$$\begin{aligned} \varphi_\zeta(1 + \kappa_q e'(x)) \\ = (1 + q^{-1} C \zeta^s x^{-1} + q^{-2} \zeta^{2s} x^{-2}) (1 + q^{-H} \zeta^s x^{-1})^{-1} (1 + q^{-H-2} \zeta^s x^{-1})^{-1}. \end{aligned}$$

Using relation (4.21) and the equation

$$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k},$$

we obtain

$$\varphi_\zeta(e_{k\delta}) = \kappa_q^{-1} (-1)^{k-1} q^{-k} \left[C_k - (q^k + q^{-k}) q^{-kH} \right] \frac{\zeta^{ks}}{k}. \quad (4.56)$$

Here the elements $C_k \in U_q(\mathfrak{sl}_2)$ are defined by the generating function

$$C(x) = \sum_{k=1}^{\infty} (-1)^{k-1} C_k \frac{x^{-k}}{k} = \log(1 + Cx^{-1} + x^{-2}). \quad (4.57)$$

In particular, we have

$$C_1 = C, \quad C_2 = -C^2 + 2, \quad C_3 = C^3 - 3C, \quad C_4 = -C^4 + 4C^2 - 2.$$

Note that all C_k belong to the center of $U_q(\mathfrak{sl}_2)$.

Below we need expressions for $\tilde{\pi}^\mu(C_k)$ and $\pi^m(C_k)$. To obtain them we apply $\tilde{\pi}^\mu$ to both sides of (4.57). Taking into account the first relation of (4.2), we see that

$$\begin{aligned} \pi^\mu(C(x)) &= \log(1 + (q^{\mu+1} + q^{-(\mu+1)})x^{-1} + x^{-2}) \\ &= \log(1 + q^{\mu+1}x^{-1}) + \log(1 + q^{-(\mu+1)}x^{-1}) \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} [q^{k(\mu+1)} + q^{-k(\mu+1)}] \frac{x^{-k}}{k}. \end{aligned}$$

Hence, we have

$$\tilde{\pi}^\mu(C_k) = q^{k(\mu+1)} + q^{-k(\mu+1)}. \quad (4.58)$$

In the same way we come to the equation

$$\pi^m(C_k) = q^{k(m+1)} + q^{-k(m+1)}. \quad (4.59)$$

To find the expressions for the images of $\mathcal{R}_{\prec\delta}$ and $\mathcal{R}_{\succ\delta}$, we again use identities (4.44). Taking into account relation (4.54) and the first equation of (4.41), we obtain

$$(\varphi_\zeta \otimes \varphi^1)(\mathcal{R}_{\prec\delta}) = 1 \otimes 1 + \kappa_q E(1 - q^{-H-1}\zeta^s)^{-1} \zeta^{s_1} \otimes E_{21}.$$

In a similar way, relation (4.55) and the second equation of (4.41) give

$$(\varphi_\zeta \otimes \varphi^1)(\mathcal{R}_{\succ\delta}) = 1 \otimes 1 + \kappa_q (1 - q^{-H-1}\zeta^s)^{-1} F \zeta^{s-s_1} \otimes E_{12}.$$

Using relations (4.43) and (4.56), we come to the equation

$$\begin{aligned} (\varphi_\zeta \otimes \varphi^1) \left(\kappa_q \sum_{k=1}^{\infty} \frac{k}{[2k]_q} e_{k\delta} \otimes f_{k\delta} \right) \\ = \left(\Lambda(q^{-1}\zeta^s) + \log(1 - q^{-H-1}\zeta^s) \right) \otimes E_{11} \\ - \left(\Lambda(q\zeta^s) + \log(1 - q^{-H+1}\zeta^s) \right) \otimes E_{22}, \end{aligned}$$

where

$$\Lambda(\zeta) = \sum_{k=1}^{\infty} \frac{1}{q^k + q^{-k}} C_k \frac{\zeta^k}{k}.$$

It is easy to see that

$$\Lambda(q\zeta) + \Lambda(q^{-1}\zeta) = -\log(1 - C\zeta + \zeta^2). \quad (4.60)$$

Hence, we have

$$\begin{aligned} (\varphi_\zeta \otimes \varphi^1)(\mathcal{R}_{\sim\delta}) = (e^{\Lambda(q^{-1}\zeta^s)} \otimes 1) \left((1 - q^{-H-1}\zeta^s) \otimes E_{11} \right. \\ \left. + (1 - C\zeta^s + \zeta^{2s})(1 - q^{-H+1}\zeta^s)^{-1} \otimes E_{22} \right). \end{aligned}$$

Finally, for the action of \mathcal{K} we have

$$\begin{aligned} (\varphi_\zeta \otimes \varphi^1)(\mathcal{K}) &\stackrel{(4.28)}{=} \sum_i \varphi_\zeta(q^{\mu_i(h_\alpha)/2}) \otimes E_{ii} \\ &\stackrel{(4.49)}{=} \varphi_\zeta(q^{h_\alpha/2}) \otimes E_{11} + \varphi_\zeta(q^{-h_\alpha/2}) \otimes E_{22} \stackrel{(4.30)}{=} q^{H/2} \otimes E_{11} + q^{-H/2} \otimes E_{22}. \end{aligned}$$

Collecting all necessary factors, we come to the expression

$$\begin{aligned} M(\zeta) = (e^{\Lambda(q^{-1}\zeta^s)} \otimes 1) \left[(q^{H/2} - q^{-1}q^{-H/2}\zeta^s) \otimes E_{11} \right. \\ \left. + \kappa_q F q^{-H/2} \zeta^{s_0} \otimes E_{12} + \kappa_q E q^{H/2} \zeta^{s_1} \otimes E_{21} \right. \\ \left. + (q^{-H/2} - q^{-1}q^{H/2}\zeta^s) \otimes E_{22} \right]. \quad (4.61) \end{aligned}$$

The corresponding matrix $\mathbb{M}(\zeta)$ has the form

$$\mathbb{M}(\zeta) = e^{\Lambda(q^{-1}\zeta^s)} \begin{pmatrix} q^{H/2} - q^{-1}q^{-H/2}\zeta^s & \kappa_q F q^{-H/2} \zeta^{s_0} \\ \kappa_q E q^{H/2} \zeta^{s_1} & q^{-H/2} - q^{-1}q^{H/2}\zeta^s \end{pmatrix}. \quad (4.62)$$

For any non-negative m we can define the monodromy operator

$$M_m(\zeta) = (\varphi_\zeta^m \otimes \varphi^1)(\mathcal{R}) = (\pi^m \otimes \text{id})(M(\zeta)),$$

where the $(m+1)$ -dimensional representation π^m of the quantum group $U_q(\mathfrak{sl}_2)$ is defined in section 4.1. Note that here we should have

$$M_1(\zeta) = (\pi^1 \otimes \text{id})(M(\zeta)) = R(\zeta), \quad (4.63)$$

where the R -operator $R(\zeta)$ is given by (4.51) or (4.53). To show this we first of all need the expression for $\pi^1(\Lambda(q^{-1}\zeta^s))$. In fact, for an arbitrary non-negative integer m equation (4.59) gives

$$\Lambda^m(\zeta) = \pi^m(\Lambda(\zeta)) = \sum_{k=1}^{\infty} \frac{q^{k(m+1)} + q^{-k(m+1)}}{q^k + q^{-k}} \frac{\zeta^k}{k}. \quad (4.64)$$

Hence, we have

$$\Lambda^m(\zeta) = \lambda_2(q^{m+1}\zeta) + \lambda_2(q^{-m-1}\zeta),$$

and, in particular,

$$\Lambda^1(q^{-1}\zeta) = \lambda_2(q\zeta) + \lambda_2(q^{-3}\zeta).$$

Using (4.36), we obtain

$$\begin{aligned} \pi^1(q^{\pm H/2}) &= q^{\pm 1/2}E_{11} + q^{\mp 1/2}E_{22}, \\ \pi^1(Fq^{-H/2}) &= q^{-1/2}E_{21}, \quad \pi^1(Eq^{H/2}) = q^{-1/2}E_{12}. \end{aligned}$$

Now, applying π^1 to both sides of (4.61) and comparing the obtained result with (4.53), we see that equation (4.63) is valid.

More generally, for any $\mu \in \mathbb{C}$ we denote

$$\tilde{M}_\mu(\zeta) = (\tilde{\varphi}_\zeta^\mu \otimes \varphi^1)(\mathcal{R}) = (\tilde{\pi}^\mu \otimes \text{id})(M(\zeta)),$$

where the infinite-dimensional representation $\tilde{\pi}^\mu$ of the quantum group $U_q(\mathfrak{sl}_2)$ is defined in section 4.1.

4.5. Transfer operators. To construct transfer operators we should first choose a twist element t . We assume that

$$t = q^{\phi h_\alpha},$$

where ϕ is a complex number. It is clear that t is a group-like element as is required. Denoting $T_m(\zeta) = T_{\varphi^m, \varphi^1}(\zeta)$, and having in mind that

$$\varphi_\zeta(t) = q^{\phi H},$$

we obtain

$$T_m(\zeta) = (\text{tr}_{V^m} \otimes \text{id})(M_m(\zeta)(\varphi_\zeta^m(t) \otimes 1)) = (\text{tr}_m \otimes \text{id})(M(\zeta)(q^{\phi H} \otimes 1)),$$

where

$$\text{tr}_m = \text{tr}_{V^m} \circ \pi^m.$$

The mapping tr_m is a trace on the algebra $U_q(\mathfrak{sl}_2)$. It is clear that in terms of the corresponding matrices $\mathbb{T}_m(\zeta) = \mathbb{T}_{\varphi^m, \varphi^1}(\zeta)$ and $\mathbb{M}(\zeta)$ we have

$$\mathbb{T}_m(\zeta) = \text{tr}_m(\mathbb{M}(\zeta) q^{\phi H}),$$

where tr_m is applied to the matrix entries. For the higher transfer matrices one obtains

$$\mathbb{T}_m(\zeta | \eta_1, \dots, \eta_n) = \text{tr}_m((\mathbb{M}(\zeta \eta_1^{-1}) \boxtimes \dots \boxtimes \mathbb{M}(\zeta \eta_n^{-1})) q^{\phi H}),$$

see relation (3.26). To find an explicit form of the transfer matrices $\mathbb{T}_m(\zeta|\eta_1, \dots, \eta_n)$ we have to know the traces for the elements of some basis of $U_q(\mathfrak{sl}_2)$. An easy calculation gives

$$\mathrm{tr}_m(E^{r+1}C^s q^{\nu H}) = 0, \quad \mathrm{tr}_m(F^{r+1}C^s q^{\nu H}) = 0, \quad (4.65)$$

$$\mathrm{tr}_m(C^s q^{\nu H}) = (q^{m+1} + q^{-m-1})^s q^{mv} \frac{1 - q^{-2\nu(m+1)}}{1 - q^{-2\nu}} = (q^{m+1} + q^{-m-1})^s [m+1]_{q^\nu}, \quad (4.66)$$

where $\nu \in \mathbb{C}$ and $r, s \in \mathbb{Z}_{\geq 0}$. For the simplest case $n = 1$ we determine that

$$\mathbb{T}_m(\zeta) = e^{\Lambda^m(q^{-1}\zeta^s)} \begin{pmatrix} [m+1]_{q^{1/2+\phi}} - [m+1]_{q^{-1/2+\phi}} q^{-1}\zeta^s & 0 \\ 0 & [m+1]_{q^{-1/2+\phi}} - [m+1]_{q^{1/2+\phi}} q^{-1}\zeta^s \end{pmatrix}.$$

This transfer matrix is finite in the limit where the twist parameter ϕ tends to zero. This is evidently true also for the higher transfer matrices $\mathbb{T}_m(\zeta|\eta_1, \dots, \eta_n)$.

The case of the infinite-dimensional representations $\tilde{\pi}^\mu$ is more subtle. Here we denote

$$\tilde{\mathrm{tr}}_\mu = \mathrm{tr}_{\tilde{V}^\mu} \circ \tilde{\pi}^\mu,$$

and for $|q^{-2\nu}| < 1$ obtain

$$\begin{aligned} \tilde{\mathrm{tr}}_\mu(E^{r+1}C^s q^{\nu H}) &= 0, & \tilde{\mathrm{tr}}_\mu(F^{r+1}C^s q^{\nu H}) &= 0, \\ \tilde{\mathrm{tr}}_\mu(C^s q^{\nu H}) &= (q^{\mu+1} + q^{-\mu-1})^s \frac{q^{\mu\nu}}{1 - q^{-2\nu}}. \end{aligned}$$

For $|q^{-2\nu}| > 1$ the trace $\tilde{\mathrm{tr}}_\mu$ is defined with the help of analytic continuation.

Again, for the simplest case we have

$$\begin{aligned} \tilde{\mathbb{T}}_\mu(\zeta) &= \tilde{\mathrm{tr}}_\mu(\mathbb{M}(\zeta) q^{\phi H}) \\ &= e^{\Lambda^\mu(q^{-1}\zeta^s)} \begin{pmatrix} \frac{q^{(1/2+\phi)\mu}}{1 - q^{-1-2\phi}} - \frac{q^{-(1/2-\phi)\mu}}{1 - q^{1-2\phi}} q^{-1}\zeta^s & 0 \\ 0 & \frac{q^{-(1/2-\phi)\mu}}{1 - q^{1-2\phi}} - \frac{q^{(1/2+\phi)\mu}}{1 - q^{-1-2\phi}} q^{-1}\zeta^s \end{pmatrix}, \end{aligned}$$

where

$$\Lambda^\mu(\zeta) = \tilde{\pi}^\mu(\Lambda(\zeta)) = \sum_{k=1}^{\infty} \frac{q^{k(\mu+1)} + q^{-k(\mu+1)}}{q^k + q^{-k}} \frac{\zeta^k}{k}.$$

The above transfer matrix is finite in the limit where the twist parameter tends to zero. This is not the case for all the higher transfer matrices

$$\tilde{\mathbb{T}}_\mu(\zeta|\eta_1, \dots, \eta_n) = \tilde{\mathrm{tr}}_\mu((\mathbb{M}(\zeta\eta_1^{-1}) \boxtimes \dots \boxtimes \mathbb{M}(\zeta\eta_n^{-1})) q^{tH}).$$

It is clear that the nonzero contributions to $\tilde{\mathbb{T}}_\mu(\zeta|\eta_1, \dots, \eta_n)$ are given by the trace of the elements

$$C^r q^{(n-2r)H/2}, \quad C^r q^{(n-2r-2)H/2}, \quad \dots, \quad C^r q^{-(n-2r)H/2},$$

where $r = 0, 1, \dots, [n/2]$. For an even n we have to take the trace of C^r , $r = 0, \dots, n/2$, and the result is evidently singular in the zero-twist limit. For an odd n there are no singularities.

It is worth to note that

$$\mathbb{T}_m(\zeta|\eta_1, \dots, \eta_n) = \tilde{\mathbb{T}}_m(\zeta|\eta_1, \dots, \eta_n) - \tilde{\mathbb{T}}_{-m-2}(\zeta|\eta_1, \dots, \eta_n)$$

and the zero-twist limit is nonsingular for the right hand side for all n . This equation suggests that we should define

$$\mathbb{T}_\mu(\zeta|\eta_1, \dots, \eta_n) = \tilde{\mathbb{T}}_\mu(\zeta|\eta_1, \dots, \eta_n) - \tilde{\mathbb{T}}_{-\mu-2}(\zeta|\eta_1, \dots, \eta_n)$$

for an arbitrary $\mu \in \mathbb{C}$. It is not difficult to prove that $\mathbb{T}_\mu(\zeta|\eta_1, \dots, \eta_n)$ is finite in the zero-twist limit for any n . One can say that $\mathbb{T}_\mu(\zeta|\eta_1, \dots, \eta_n)$ is the transfer matrix defined by the trace

$$\mathrm{tr}_\mu = \tilde{\mathrm{tr}}_\mu - \tilde{\mathrm{tr}}_{-\mu-2} \quad (4.67)$$

on the algebra $U_q(\mathfrak{sl}_2)$. It is instructive to compare the definition of tr_μ with the definition of trace given in the papers [22, 24], see also [13, 23, 57] for the limiting case $q = 1$.

4.6. L-operators. Remind that with the definition of a quantum group used by us, the quasitriangularity is understood in some restricted sense, see section 4.2. We can only define the action of the universal R -matrix in the tensor product of weight representations. It is easy to see that to determine the action of the universal R -matrix (4.23) on the tensor product of two representation spaces, it suffices to use for the first factor representations of the subalgebra $U_q(\mathfrak{b}_+)$ and for the second one representations of the subalgebra $U_q(\mathfrak{b}_-)$. Here the Borel subalgebra $U_q(\mathfrak{b}_+)$ is generated by e_0, e_1 and $q^x, x \in \tilde{\mathfrak{h}}$, and the Borel subalgebra $U_q(\mathfrak{b}_-)$ is generated by f_0, f_1 and $q^x, x \in \tilde{\mathfrak{h}}$.

It is clear that any representation of the algebra $U_q(\mathcal{L}(\mathfrak{sl}_2))$ generates representations of the subalgebras $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$. However, this does not give new objects. There are other methods to construct representations of $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ from representations of $U_q(\mathcal{L}(\mathfrak{sl}_2))$. We restrict ourselves by the case of the Borel subalgebra $U_q(\mathfrak{b}_+)$.

First note that if φ is a representation of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ and $\xi \in \tilde{\mathfrak{h}}^*$, then the mapping $\varphi[\xi]$ defined by the equations

$$\varphi[\xi](e_i) = \varphi(e_i), \quad \varphi[\xi](q^x) = q^{\xi(x)} \varphi(q^x)$$

is a representation of $U_q(\mathfrak{b}_+)$ called a *shifted representation*. It follows from (4.11) that we have to assume that

$$\xi(h_{\delta-\alpha}) = -\xi(h_\alpha).$$

Taking into account relation (4.27), we see that for any weight representations φ and ψ one has

$$(\varphi[\xi] \otimes \psi)(\mathcal{K}) = (\varphi \otimes \psi)(\mathcal{K})(1 \otimes \psi(q^{\xi(h_\alpha)h_\alpha/2})).$$

Therefore, for the universal R -matrix we can write

$$(\varphi[\xi] \otimes \psi)(\mathcal{R}) = (\varphi \otimes \psi)(\mathcal{R})(1 \otimes \psi(q^{\xi(h_\alpha)h_\alpha/2})).$$

In fact, we can even write

$$(\varphi[\xi] \otimes \mathrm{id})(\mathcal{R}) = (\varphi \otimes \mathrm{id})(\mathcal{R})(1 \otimes q^{\xi(h_\alpha)h_\alpha/2}),$$

having in mind that this equation is true only for weight representations. Under the same assumption, we have for the universal monodromy matrix the equation

$$\mathcal{M}_{\varphi[\xi]}(\zeta) = \mathcal{M}_\varphi(\zeta)(1 \otimes q^{\xi(h_\alpha)h_\alpha/2}),$$

and for the universal transfer matrix the equation

$$\mathcal{T}_{\varphi[\xi]}(\zeta) = \mathcal{T}_\varphi(\zeta) q^{\xi(h_\alpha)(h_\alpha+2\phi)/2}. \quad (4.68)$$

Thus, the use of shifted representations does not give anything really new. Nevertheless, we meet universal transfer matrices corresponding to shifted representations when proving the functional relations.

Now let us describe how to obtain the representation necessary for the construction of L -operators. Starting with the representation $\tilde{\varphi}_\zeta^\mu$ of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ defined by equations (4.33)–(4.35), we obtain the shifted representation $\tilde{\varphi}_\zeta^\mu[\zeta]$ of $U_q(\mathfrak{b}_+)$ defined by the relations

$$q^{vh_\alpha}v_n = q^{\nu(\mu-2n+\zeta(h_\alpha))}v_n, \quad q^{\nu h_{\delta-\alpha}}v_n = q^{-\nu(\mu-2n-\zeta(h_{\delta-\alpha}))}v_n, \quad (4.69)$$

$$e_\alpha v_n = \zeta^{s_1}[n]_q[\mu-n+1]_q v_{n-1}, \quad e_{\delta-\alpha}v_n = \zeta^{s_0}v_{n+1}, \quad (4.70)$$

We denote the corresponding $U_q(\mathfrak{b}_+)$ -module by $\tilde{V}_\zeta^\mu[\zeta]$. Assume that

$$\zeta(h_\alpha) = -\zeta(h_{\delta-\alpha}) = -\mu.$$

Relations (4.69) take the form

$$q^{vh_\alpha}v_n = q^{-2\nu n}v_n, \quad q^{\nu h_{\delta-\alpha}}v_n = q^{2\nu n}v_n. \quad (4.71)$$

Note that we can multiply the operators corresponding to the generators e_0 and e_1 by arbitrary nonzero complex numbers. This again gives a representation of $U_q(\mathfrak{b}_+)$. Represent the first relation of (4.70) as

$$q^{-\mu-1}e_\alpha v_n = \zeta^{s_1}\kappa_q^{-1}(q^{-n} - q^{-2\mu+n-2})[n]_q v_{n-1}.$$

Now we rescale the operator corresponding to e_α as $e_\alpha \rightarrow q^{\mu+1}e_\alpha$ and consider the limit $\mu \rightarrow \infty$ along the real axis. This gives instead of (4.70) the relations

$$e_\alpha v_n = \zeta^{s_1}\kappa_q^{-1}q^{-n}[n]_q v_{n-1}, \quad e_{\delta-\alpha}v_n = \zeta^{s_0}v_{n+1}. \quad (4.72)$$

Relations (4.71) and (4.72) define a representation of $U_q(\mathfrak{b}_+)$. Note that this representation cannot be extended to a representation of the full quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$. It is useful to give an interpretation of (4.71) and (4.72) in terms of q -oscillators. Let us remind the necessary definitions, see, for example the book [46].

Let \hbar be a complex number such that $q = \exp \hbar \neq 0, \pm 1$. The q -oscillator algebra Osc_q is a unital associative \mathbb{C} -algebra with generators $b^\dagger, b, q^{\nu N}, \nu \in \mathbb{C}$, and relations

$$\begin{aligned} q^0 &= 1, & q^{\nu_1 N}q^{\nu_2 N} &= q^{(\nu_1+\nu_2)N}, \\ q^{\nu N}b^\dagger q^{-\nu N} &= q^\nu b^\dagger, & q^{\nu N}b q^{-\nu N} &= q^{-\nu}b, \\ b^\dagger b &= \kappa_q^{-1}(q^N - q^{-N}), & bb^\dagger &= \kappa_q^{-1}(q^{N+1} - q^{-N-1}). \end{aligned}$$

There are two interesting for us representations of Osc_q . First, let W^+ be a free vector space generated by the set $\{v_0, v_1, \dots\}$. One can show that the relations

$$q^{\nu N}v_n = q^{\nu n}v_n, \quad (4.73)$$

$$b^\dagger v_n = v_{n+1}, \quad b v_n = [n]_q v_{n-1}, \quad (4.74)$$

where we assume that $v_{-1} = 0$, endow W^+ with the structure of an Osc_q -module. We denote the corresponding representation of the algebra Osc_q by χ^+ . Further, let W^- be a free vector space generated by the set $\{u_0, u_1, \dots\}$. The relations

$$q^{\nu N}u_n = q^{-\nu(n+1)}u_n, \quad (4.75)$$

$$b u_n = u_{n+1}, \quad b^\dagger u_n = -[n]_q u_{n-1}, \quad (4.76)$$

where we assume that $u_{-1} = 0$, endow W^- with the structure of an Osc_q -module. We denote the corresponding representation of Osc_q by χ^- .

Return again to relations (4.71) and (4.72). Assume that the operators N , b^\dagger and b act in the representation space in accordance with (4.73) and (4.74). This allows to write (4.71) and (4.72) as

$$\begin{aligned} q^{vh_\alpha} v_n &= q^{-2\nu N} v_n, & q^{vh_{\delta-\alpha}} v_n &= q^{2\nu N} v_n, \\ e_\alpha v_n &= \zeta^{s_1} \kappa_q^{-1} b q^{-N} v_n, & e_{\delta-\alpha} v_n &= \zeta^{s_0} b^\dagger v_n. \end{aligned}$$

These equations suggest to us a homomorphism $\rho : U_q(\mathfrak{b}_+) \rightarrow \text{Osc}_q$ defined by

$$\rho(q^{vh_\alpha}) = q^{-2\nu N}, \quad \rho(q^{vh_{\delta-\alpha}}) = q^{2\nu N}, \quad (4.77)$$

$$\rho(e_\alpha) = \kappa_q^{-1} b q^{-N}, \quad \rho(e_{\delta-\alpha}) = b^\dagger, \quad (4.78)$$

and the homomorphisms $\rho_\zeta : U_q(\mathfrak{b}_+) \rightarrow \text{Osc}_q$, $\zeta \in \mathbb{C}^\times$, as $\rho_\zeta = \rho \circ \Phi_\zeta$. We can now define the representations

$$\rho^\pm = \chi^\pm \circ \rho, \quad \rho_\zeta^\pm = \chi^\pm \circ \rho_\zeta \quad (4.79)$$

of the Borel subalgebra $U_q(\mathfrak{b}_+)$. For the representations ρ_ζ^\pm we have explicitly

$$\rho_\zeta^\pm(q^{vh_\alpha}) = \chi^\pm(q^{-2\nu N}), \quad \rho_\zeta^\pm(q^{vh_{\delta-\alpha}}) = \chi^\pm(q^{2\nu N}), \quad (4.80)$$

$$\rho_\zeta^\pm(e_\alpha) = \zeta^{s_1} \kappa_q^{-1} \chi^\pm(b q^{-N}), \quad \rho_\zeta^\pm(e_{\delta-\alpha}) = \zeta^{s_0} \chi^\pm(b^\dagger). \quad (4.81)$$

We denote the $U_q(\mathfrak{b}_+)$ -modules corresponding to the representations ρ_ζ^\pm by W_ζ^\pm .

Let us construct the L -operator

$$L_{\rho,\psi}(\zeta) = L_{\rho,\psi}(\zeta|1) = (\rho_\zeta \otimes \psi)(\mathcal{R})$$

choosing as ρ the homomorphism defined by (4.77) and (4.78), and as ψ the representation φ^1 . As for the case of monodromy operators, we extend the notion of L -operators allowing for using general homomorphisms instead of representations. To simplify notation we write instead of $L_{\rho,\psi}(\zeta)$ just $L(\zeta)$.

Having in mind (4.28), (4.37) and (4.77), we observe that

$$(\rho_\zeta \otimes \varphi^1)(\mathcal{K}) = q^{-N} \otimes E_{11} + q^N \otimes E_{22}. \quad (4.82)$$

Further, one can easily determine that definition (4.13) together with (4.78) gives

$$\rho_\zeta(e'_\delta) = \kappa_q^{-1} q^{-1} \zeta^s, \quad (4.83)$$

and, using (4.14) and (4.15), we immediately obtain

$$\rho_\zeta(e_{\alpha+k\delta}) = 0, \quad \rho_\zeta(e_{(\delta-\alpha)+k\delta}) = 0, \quad k \geq 1. \quad (4.84)$$

Taking into account (4.39) and (4.44), we come to

$$(\rho_\zeta \otimes \varphi^1)(\mathcal{R}_{\prec\delta}) = 1 \otimes 1 + b q^{-N} \zeta^{s_1} \otimes E_{21}, \quad (4.85)$$

$$(\rho_\zeta \otimes \varphi^1)(\mathcal{R}_{\succ\delta}) = 1 \otimes 1 + \kappa_q b^\dagger \zeta^{s-s_1} \otimes E_{12}. \quad (4.86)$$

Definition (4.16) and equations (4.84) give

$$\rho_\zeta(e'_{k\delta}) = 0, \quad k > 1,$$

and one easily finds that

$$\rho_\zeta(e_{k\delta}) = (-1)^{k-1} \kappa_q^{-1} q^{-k} \frac{\zeta^{ks}}{k}.$$

Now, using relations (4.24) and (4.43), we obtain

$$(\rho_\zeta \otimes \varphi^1)(\mathcal{R}_{\sim\delta}) = e^{\lambda_2(q^{-1}\zeta^s)} [1 \otimes E_{11} + (1 - \zeta^s) \otimes E_{22}], \quad (4.87)$$

where the function $\lambda_2(\zeta)$ is defined by (4.48).

Multiplying expressions (4.82), (4.85), (4.86) and (4.87) in the prescribed order, we come to the following L -operator:

$$L(\zeta) = e^{\lambda_2(q^{-1}\zeta^s)} [q^{-N} \otimes E_{11} + \kappa_q b^\dagger q^N \zeta^{s-s_1} \otimes E_{12} + b q^{-2N} \zeta^{s_1} \otimes E_{21} + (q^N - q^{-2} q^{-N} \zeta^s) \otimes E_{22}].$$

In the matrix form it looks as

$$\mathbb{L}(\zeta) = e^{\lambda_2(q^{-1}\zeta^s)} \begin{pmatrix} q^{-N} & \kappa_q b^\dagger q^N \zeta^{s-s_1} \\ b q^{-2N} \zeta^{s_1} & q^N - q^{-2} q^{-N} \zeta^s \end{pmatrix}.$$

It is evident that the relations

$$\begin{aligned} \sigma(h_0) &= h_1, & \sigma(h_1) &= h_0, \\ \sigma(e_0) &= e_1, & \sigma(e_1) &= e_0, & \sigma(f_0) &= f_1, & \sigma(f_1) &= f_0 \end{aligned}$$

define an automorphism of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ and, via the restriction, an automorphism of $U_q(\mathfrak{b}_+)$. Therefore, the mappings

$$\bar{\rho} = \rho \circ \sigma, \quad \bar{\rho}_\zeta = \bar{\rho} \circ \Phi_\zeta$$

are homomorphisms from $U_q(\mathfrak{b}_+)$ to Osc_q , and the mappings

$$\bar{\rho}^\pm = \chi^\pm \circ \bar{\rho}, \quad \bar{\rho}_\zeta^\pm = \chi^\pm \circ \bar{\rho}_\zeta \quad (4.88)$$

are representations of $U_q(\mathfrak{b}_+)$. We denote the $U_q(\mathfrak{b}_+)$ -modules corresponding to the representations $\bar{\rho}_\zeta^\pm$ by \bar{W}_ζ^\pm .

Let us find the expression for the L -operator

$$\bar{L}(\zeta) = L_{\bar{\rho}, \varphi^1}(\zeta).$$

Calculations give

$$\begin{aligned} \bar{\rho}_\zeta(e_{\alpha+k\delta}) &= (-1)^k q^{-k} q^{-2kN} b^\dagger \zeta^{s_1+ks}, \\ \bar{\rho}_\zeta(e_{(\delta-\alpha)+k\delta}) &= \kappa_q^{-1} (-1)^k q^{-k} b q^{-(2k+1)N} \zeta^{s_0+ks}, \\ \bar{\rho}_\zeta(e_{k\delta}) &= \kappa_q^{-1} (-1)^k q^{-k} [(1 + q^{-2k}) q^{-2kN} - 1] \frac{\zeta^{ks}}{k}. \end{aligned}$$

Using these equations, we obtain

$$\begin{aligned} (\bar{\rho}_\zeta \otimes \varphi^1)(\mathcal{R}_{\prec\delta}) &= 1 \otimes 1 + \kappa_q b^\dagger (1 - q^{-2} q^{-2N} \zeta^s)^{-1} \zeta^{s_1} \otimes E_{21}, \\ (\bar{\rho}_\zeta \otimes \varphi^1)(\mathcal{R}_{\sim\delta}) &= e^{\lambda_2(q^{-1}\zeta^s)} [(1 - q^{-2} q^{-2N} \zeta^s) \otimes E_{11} + (1 - \zeta^s) (1 - q^{-2N} \zeta^s)^{-1} \otimes E_{22}], \\ (\bar{\rho}_\zeta \otimes \varphi^1)(\mathcal{R}_{\succ\delta}) &= 1 \otimes 1 + b q^{-N} (1 - q^{-2N} \zeta^s)^{-1} \zeta^{s-s_1} \otimes E_{12}, \\ (\bar{\rho}_\zeta \otimes \varphi^1)(\mathcal{K}) &= q^N \otimes E_{11} + q^{-N} \otimes E_{22} \end{aligned}$$

Multiplying these expressions in the order prescribed by (4.23), we determine that

$$\begin{aligned} \bar{L}(\zeta) &= e^{\lambda_2(q^{-1}\zeta^s)} [(q^N - q^{-2} q^{-N} \zeta^s) \otimes E_{11} \\ &\quad + b q^{-2N} \zeta^{s-s_1} \otimes E_{12} + \kappa_q b^\dagger q^N \zeta^{s_1} \otimes E_{21} + q^{-N} \otimes E_{22}], \end{aligned}$$

or in the matrix form

$$\bar{\mathbb{L}}(\zeta) = e^{\lambda_2(q^{-1}\zeta^s)} \begin{pmatrix} q^N - q^{-2}q^{-N}\zeta^s & b q^{-2N}\zeta^{s-s_1} \\ \kappa_q b^+ q^N \zeta^{s_1} & q^{-N} \end{pmatrix}.$$

4.7. Q-operators. We again use the twist element

$$t = q^{\phi h_\alpha},$$

where ϕ is a complex number. Using the representations χ^+ and χ^- , we can define two traces

$$\mathrm{tr}_+ = \mathrm{tr}_{W^+} \circ \chi^+, \quad \mathrm{tr}_- = \mathrm{tr}_{W^-} \circ \chi^-$$

on the algebra Osc_q . In fact, these traces differ only by an overall sign. We introduce two Q-operators,

$$Q'(\zeta) = Q_{\rho^+, \varphi^1}(\zeta), \quad \bar{Q}'(\zeta) = Q_{\bar{\rho}^-, \varphi^1}(\zeta).$$

We use the prime here because we slightly redefine the Q-operators below. Taking into account that

$$\rho(t) = q^{-2\phi N}, \quad \bar{\rho}(t) = q^{2\phi N},$$

we obtain

$$Q'(\zeta) = (\mathrm{tr}_+ \otimes \mathrm{id})(L(\zeta)(q^{-2\phi N} \otimes 1)), \quad \bar{Q}'(\zeta) = (\mathrm{tr}_- \otimes \mathrm{id})(\bar{L}(\zeta)(q^{2\phi N} \otimes 1)).$$

In terms of the corresponding matrices we have

$$Q'(\zeta) = \mathrm{tr}_+(\mathbb{L}(\zeta) q^{-2\phi N}), \quad \bar{Q}'(\zeta) = \mathrm{tr}_-(\bar{\mathbb{L}}(\zeta) q^{2\phi N}),$$

where tr_+ is applied to the matrix entries, and, in general,

$$Q'(\zeta|\eta_1, \dots, \eta_n) = \mathrm{tr}_+((\mathbb{L}(\zeta\eta_1^{-1}) \boxtimes \dots \boxtimes \mathbb{L}(\zeta\eta_n^{-1}))q^{-2\phi N}),$$

$$\bar{Q}'(\zeta|\eta_1, \dots, \eta_n) = \mathrm{tr}_-((\bar{\mathbb{L}}(\zeta\eta_1^{-1}) \boxtimes \dots \boxtimes \bar{\mathbb{L}}(\zeta\eta_n^{-1}))q^{2\phi N}).$$

Using (4.73) and (4.74), we see that for $|q| < 1$ one has

$$\mathrm{tr}_+(q^{\nu N}) = \frac{1}{1 - q^\nu},$$

$$\mathrm{tr}_+((b^+)^{r+1}q^{\nu N}) = 0, \quad \mathrm{tr}_+(b^{r+1}q^{\nu N}) = 0$$

for any $\nu \in \mathbb{C}$ and $r \in \mathbb{Z}_{\geq 0}$. For $|q| > 1$ we define the trace tr_+ by analytic continuation. Using the above relations, for $n = 1$ we obtain

$$Q'(\zeta) = e^{\lambda_2(q^{-1}\zeta^s)} \begin{pmatrix} \frac{1}{1 - q^{-1-2\phi}} & 0 \\ 0 & \frac{1}{1 - q^{1-2\phi}} - \frac{1}{1 - q^{-1-2\phi}} q^{-2}\zeta^s \end{pmatrix}$$

and

$$\bar{Q}'(\zeta) = -e^{\lambda_2(q^{-1}\zeta^s)} \begin{pmatrix} \frac{1}{1 - q^{1+2\phi}} - \frac{1}{1 - q^{-1+2\phi}} q^{-2}\zeta^s & 0 \\ 0 & \frac{1}{1 - q^{-1+2\phi}} \end{pmatrix}.$$

As well as for the higher transfer matrices $\tilde{\mathbb{T}}_\mu(\zeta|\eta_1, \dots, \eta_n)$, one sees that the matrices $Q'(\zeta|\eta_1, \dots, \eta_n)$ and $\bar{Q}'(\zeta|\eta_1, \dots, \eta_n)$ are finite in the zero-twist limit for an odd n , and singular for an even n .

5. FUNCTIONAL RELATIONS FOR $U_q(\mathcal{L}(\mathfrak{sl}_2))$

In this section we derive certain relations satisfied by the universal transfer operators

$$\tilde{\mathcal{T}}_\mu(\zeta) = \mathcal{T}_{\tilde{\varphi}^\mu}(\zeta), \quad \mathcal{T}_m(\zeta) = \mathcal{T}_{\varphi^m}(\zeta),$$

and the universal Q -operators

$$\mathcal{Q}'(\zeta) = \mathcal{Q}_{\rho^+}(\zeta), \quad \overline{\mathcal{Q}}'(\zeta) = \mathcal{Q}_{\bar{\rho}^-}(\zeta),$$

where the representations $\tilde{\varphi}^\mu$, φ^m of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ are defined in section 4.3, and the representations ρ^+ , $\bar{\rho}^-$ of $U_q(\mathfrak{b}_+)$ in section 4.6. These relations, known as functional relations, appear to be very useful for investigation of the corresponding integrable systems.

There are relations which are due only to the fact that the universal transfer operators and universal Q -operators are constructed from the universal R -matrices. Another set of relations depends on the structure of the representations used for their construction. Here, to analyze the products of the operators, we should analyze the tensor products of the corresponding representations, see, for example, relations (3.30) and (3.31).

5.1. Tensor product of representations.

5.1.1. *Tensor product of representations $\rho_{\zeta_1}^+$ and $\rho_{\zeta_2}^+$.* To analyze the product of the universal Q -operators $\mathcal{Q}'(\zeta_1)$ and $\mathcal{Q}'(\zeta_2)$ we consider the tensor product of the representations $\rho_{\zeta_1}^+$ and $\rho_{\zeta_2}^+$. Here the representation space is $W^+ \otimes W^+$, which is also the representation space of the representation $\chi^+ \otimes \chi^+$ of the algebra $\text{Osc}_q \otimes \text{Osc}_q$.

It is not difficult to see that⁹

$$\begin{aligned} (\rho_{\zeta_1}^+ \otimes_{\Delta} \rho_{\zeta_2}^+)(q^{vh_0}) &= (\rho_{\zeta_1}^+ \otimes \rho_{\zeta_2}^+)(\Delta(q^{vh_0})) \stackrel{(4.9)}{=} (\rho_{\zeta_1}^+ \otimes \rho_{\zeta_2}^+)(q^{vh_0} \otimes q^{vh_0}) \\ &\stackrel{(4.77)}{=} \chi^+(q^{2vN}) \otimes \chi^+(q^{2vN}) = (\chi^+ \otimes \chi^+)(q^{2vN} \otimes q^{2vN}). \end{aligned}$$

Similarly, we obtain

$$(\rho_{\zeta_1}^+ \otimes_{\Delta} \rho_{\zeta_2}^+)(q^{vh_1}) = (\chi^+ \otimes \chi^+)(q^{-2vN} \otimes q^{-2vN}).$$

Below we denote

$$\begin{aligned} q^{\mu N_A + \nu N_B} &= q^{\mu N} \otimes q^{\nu N}, \quad \mu, \nu \in \mathbb{C}, \\ b_A &= b \otimes 1, \quad b_B = 1 \otimes b, \quad b_A^\dagger = b^\dagger \otimes 1, \quad b_B^\dagger = 1 \otimes b^\dagger. \end{aligned}$$

Now, using module notation, we can write

$$q^{vh_0} w = q^{2v(N_A + N_B)} w, \quad q^{vh_1} w = q^{-2v(N_A + N_B)} w$$

for any $w \in W^+ \otimes W^+$. Further, we obtain

$$e_0 w = (b_A^\dagger \zeta_1^{s_0} + b_B^\dagger q^{-2N_A} \zeta_2^{s_0}) w, \tag{5.1}$$

$$e_1 w = \kappa_q^{-1} (b_A q^{-N_A} \zeta_1^{s_1} + b_B q^{2N_A - N_B} \zeta_2^{s_1}) w. \tag{5.2}$$

⁹Remind that $h_0 = h_{\delta-\alpha}$, $h_1 = h_\alpha$, $e_0 = e_{\delta-\alpha}$ and $e_1 = e_\alpha$.

Remind that W^+ is the free vector space generated by the set $\{v_n\}_{n \in \mathbb{Z}_{\geq 0}}$. The vectors $v_n \otimes v_k$, $n, k \in \mathbb{Z}_{\geq 0}$, form a basis of $W^+ \otimes W^+$. Consider another basis $\{w_{n,k}\}_{n,k \in \mathbb{Z}_{\geq 0}}$, where

$$w_{n,k} = (e_0)^n (\zeta_2^{s_1} b_A^\dagger)^k (v_0 \otimes v_0).$$

Let us show that

$$q^{vh_0} w_{n,k} = q^{2v(n+k)} w_{n,k}, \quad (5.3)$$

$$q^{vh_1} w_{n,k} = q^{-2v(n+k)} w_{n,k}, \quad (5.4)$$

$$e_0 w_{n,k} = w_{n+1,k}, \quad (5.5)$$

$$\begin{aligned} e_1 w_{n,k} &= \kappa_q^{-1} q^{-2n-k} [k]_q (\zeta_1 \zeta_2)^{s_1} w_{n,k-1} \\ &\quad + \kappa_q^{-1} q^{-n} [n]_q (\zeta_1^s + \zeta_2^s) w_{n-1,k} + q^{-1} [n]_q [n-1]_q (\zeta_1 \zeta_2)^{s_0} w_{n-2,k+1}. \end{aligned} \quad (5.6)$$

In fact, the first three equations are evident, and one should prove only the last one. To this end, we move e_1 through e_0 and introduce some operators arising during this process. Then we move these operators through the remaining factors e_0 , and so on. The process terminates when we arrive at an operator which can be moved through e_0 without introducing new operators. To finish, we determine the action of e_1 and all new operators on the vectors of the form $w_{0,k}$.

We start with defining an operator x by the equation

$$x w = (e_1 e_0 - q^{-2} e_0 e_1) w \quad (5.7)$$

for any $w \in W^+ \otimes W^+$. Explicitly, we have

$$x w = \left[\kappa_q^{-1} q^{-1} (\zeta_1^s + \zeta_2^s) + (q + q^{-1}) \zeta_1^{s_0} \zeta_2^{s_1} b_A^\dagger b_B q^{2N_A - N_B} \right] w. \quad (5.8)$$

Now we move x and introduce the operator y as

$$y w = (x e_0 - e_0 x) w, \quad (5.9)$$

or, explicitly,

$$y w = \left[q^{-1} (q + q^{-1}) \zeta_1^{s_0} \zeta_2^s b_A^\dagger + \kappa_q q (q + q^{-1}) \zeta_1^{2s_0} \zeta_2^{s_1} (b_A^\dagger)^2 b_B q^{2N_A - N_B} \right] w. \quad (5.10)$$

One can verify that

$$(y e_0 - q^2 e_0 y) w = 0. \quad (5.11)$$

In fact, this equation is a consequence of the Serre relations (4.7) and its validity does not depend on the used representation of the quantum group $U_q(\mathfrak{b}_+)$. Equations (5.7), (5.9) and (5.11) give

$$e_1 (e_0)^n w = \left[q^{-2n} (e_0)^n e_1 + q^{-n+1} [n]_q (e_0)^{n-1} x + (q + q^{-1})^{-1} [n]_q [n-1]_q (e_0)^{n-2} y \right] w,$$

and we obtain

$$\begin{aligned} e_1 w_{n,k} &= e_1 (e_0)^n w_{0,k} = \left[q^{-2n} (e_0)^n e_1 + q^{-n+1} [n]_q (e_0)^{n-1} x \right. \\ &\quad \left. + (q + q^{-1})^{-1} [n]_q [n-1]_q (e_0)^{n-2} y \right] w_{0,k}. \end{aligned} \quad (5.12)$$

Using the explicit relations (5.2), (5.8) and (5.10), we see that

$$e_1 w_{0,k} = \kappa_q^{-1} q^{-k} [k]_q (\zeta_1 \zeta_2)^{s_1} w_{0,k-1},$$

$$x w_{0,k} = \kappa_q^{-1} q^{-1} (\zeta_1^s + \zeta_2^s) w_{0,k},$$

$$y w_{0,k} = q^{-1}(q + q^{-1}) (\zeta_1 \zeta_2)^{s_0} w_{0,k+1}.$$

Now, it is easy to see that equation (5.6) is true.

5.1.2. *Tensor product of representations $\overline{\rho}_{\zeta_1}$ and $\overline{\rho}_{\zeta_2}$.* To analyse the product of the universal Q -operators $\overline{Q}'(\zeta_1)$ and $\overline{Q}'(\zeta_2)$, we use the tensor product of the representations $\overline{\rho}_{\zeta_1}$ and $\overline{\rho}_{\zeta_2}$. Here we obtain that

$$q^{vh_0} w = q^{-2v(N_A+N_B)} w, \quad q^{vh_1} w = q^{2v(N_A+N_B)} w,$$

and that

$$e_0 w = \kappa_q^{-1} (b_A q^{-N_A} \zeta_1^{s_0} + b_B q^{2N_A-N_B} \zeta_2^{s_0}) w, \quad (5.13)$$

$$e_1 w = (b_A^\dagger \zeta_1^{s_1} + b_B^\dagger q^{-2N_A} \zeta_2^{s_1}) w \quad (5.14)$$

for any $w \in \overline{W}^- \otimes \overline{W}^-$. Introduce now a basis

$$w_{n,k} = (e_0)^n (\zeta_1^{s_1} b_B)^\dagger (u_0 \otimes u_0).$$

and show that

$$q^{vh_0} w_{n,k} = q^{2v(n+k+2)} w_{n,k}, \quad (5.15)$$

$$q^{vh_1} w_{n,k} = q^{-2v(n+k+2)} w_{n,k}, \quad (5.16)$$

$$e_0 w_{n,k} = w_{n+1,k}, \quad (5.17)$$

$$e_1 w_{n,k} = -q^{2n+2} [k]_q (\zeta_1 \zeta_2)^{s_1} w_{n,k-1} \\ - \kappa_q^{-1} q^n [n]_q (\zeta_1^s + \zeta_2^s) w_{n-1,k} + \kappa_q^{-1} q^k [n]_q [n-1]_q (\zeta_1 \zeta_2)^{s_0} w_{n-2,k+1}. \quad (5.18)$$

The first three equations are evident, and to prove the fourth one we introduce the operators x and y by

$$x w = (e_1 e_0 - q^2 e_0 e_1) w \quad (5.19)$$

and (5.9). Explicitly, we have

$$x w = - \left[\kappa_q^{-1} q (\zeta_1^s + \zeta_2^s) + q^2 (q + q^{-1}) \zeta_1^{s_1} \zeta_2^{s_0} b_A^\dagger b_B q^{2N_A-N_B} \right] w$$

and

$$y w = \left[\kappa_q^{-1} q (q + q^{-1}) \zeta_1^s \zeta_2^{s_0} b_B q^{2N_A-N_B} + q^4 (q + q^{-1}) \zeta_1^{s_1} \zeta_2^{s_0} b_A^\dagger (b_B)^2 q^{4N_A-2N_B} \right] w.$$

It follows from these relations and from (5.14) that

$$e_1 w_{0,k} = -q^2 [k]_q (\zeta_1 \zeta_2)^{s_1} w_{0,k-1},$$

$$x w_{0,k} = -\kappa_q^{-1} q (\zeta_1^s + \zeta_2^s) w_{0,k},$$

$$y w_{0,k} = \kappa_q^{-1} q^k (q + q^{-1}) (\zeta_1 \zeta_2)^{s_0} w_{0,k+1}.$$

Now, instead of (5.12), we have

$$e_1 w_{n,k} = e_1 (e_0)^n w_{0,k} = \left[q^{2n} (e_0)^n e_1 + q^{n-1} [n]_q (e_0)^{n-1} x \right. \\ \left. + (q + q^{-1})^{-1} [n]_q [n-1]_q (e_0)^{n-2} y \right] w_{0,k}. \quad (5.20)$$

Using this equation, we conclude that equation (5.18) is true.

5.1.3. *Tensor product of representations* $\rho_{\zeta_1}^+$ and $\bar{\rho}_{\zeta_2}^-$. Now we consider the tensor product of the representations $\rho_{\zeta_1}^+$ and $\bar{\rho}_{\zeta_2}^-$ which is necessary to analyse the product of the universal Q -operators $\mathcal{Q}'(\zeta_1)$ and $\overline{\mathcal{Q}}'(\zeta_2)$. For this tensor product we obtain that

$$q^{vh_0} w = q^{2\nu(N_A - N_B)} w, \quad q^{vh_1} w = q^{-2\nu(N_A - N_B)} w,$$

and that

$$e_0 w = (\zeta_1^{s_0} b_A^\dagger + \kappa_q^{-1} \zeta_2^{s_0} b_B q^{-2N_A - N_B}) w, \quad (5.21)$$

$$e_1 w = (\kappa_q^{-1} \zeta_1^{s_1} b_A q^{-N_A} + \zeta_2^{s_1} b_B^\dagger q^{2N_A}) w. \quad (5.22)$$

A convenient basis of $W^+ \otimes \overline{W}^-$ is formed by the vectors

$$w_{n,k} = ((\zeta_1 \zeta_2)^{-s_0/2} e_0)^n (\zeta_2^{-s_1} b_B)^k (v_0 \otimes u_0).$$

Here one obtains

$$q^{vh_0} w_{n,k} = q^{2\nu(n+k+1)} w_{n,k}, \quad (5.23)$$

$$q^{vh_1} w_{n,k} = q^{-2\nu(n+k+1)} w_{n,k}, \quad (5.24)$$

$$e_0 w_{n,k} = (\zeta_1 \zeta_2)^{s_0/2} w_{n+1,k}, \quad (5.25)$$

$$e_1 w_{n,k} = -q^{2n} [k]_q w_{n,k-1} + \kappa_q^{-1} [n]_q (\zeta_1 \zeta_2)^{-s_0/2} q^{-n} (\zeta_1^{s_0+s_1} - q^{2n} \zeta_2^{s_0+s_1}) w_{n-1,k}. \quad (5.26)$$

Let us prove the last equation. The operators x and y defined by relations (5.19) and (5.9), act on a vector w of $W^+ \otimes \overline{W}^-$ as

$$\begin{aligned} x w = & \left[\kappa_q^{-1} (q + q^{-1}) \zeta_1^s q^{-2N_A} \right. \\ & \left. - \kappa_q^{-1} q (\zeta_1^s + \zeta_2^s) - \kappa_q^{-1} (q + q^{-1}) q^2 \zeta_1^{s_1} \zeta_2^{s_0} b_A b_B q^{-3N_A - N_B} \right] w \end{aligned}$$

and

$$\begin{aligned} y w = & - \left[q^{-1} (q + q^{-1}) \zeta_1^{s+s_0} b_A^\dagger q^{-2N_A} \right. \\ & + \kappa_q^{-1} (q + q^{-1})^2 \zeta_1^s \zeta_2^{s_0} b_B q^{-4N_A - N_B} - \kappa_q^{-1} q (q + q^{-1}) \zeta_1^s \zeta_2^{s_0} b_B q^{-2N_A - N_B} \\ & \left. - \kappa_q^{-1} q^4 (q + q^{-1}) \zeta_1^{s_1} \zeta_2^{2s_0} b_A (b_B)^2 q^{-5N_A - 2N_B} \right] w. \end{aligned}$$

It follows from these equations and from (5.22) that

$$\begin{aligned} e_1 w_{0,k} &= -[k]_q w_{0,k-1}, \\ x w_{0,k} &= \kappa_q^{-1} q^{-1} (\zeta_1^s - q^2 \zeta_2^s) w_{0,k}, \\ y w_{0,k} &= -q^{-1} (q + q^{-1}) \zeta_1^s e_0 w_{0,k}. \end{aligned}$$

Using relation (5.20), we see that equation (5.26) is true.

Introduce the parameters ζ and μ such that

$$\zeta = (\zeta_1 \zeta_2)^{1/2}, \quad q^{\mu+1} = (\zeta_1 / \zeta_2)^{(s_0+s_1)/2}. \quad (5.27)$$

The inverse transformation to the parameters ζ_1 and ζ_2 is

$$\zeta_1 = q^{(\mu+1)/s} \zeta, \quad \zeta_2 = q^{-(\mu+1)/s} \zeta.$$

In terms of the new parameters equations (5.25) and (5.26) take the form

$$\begin{aligned} e_0 w_{n,k} &= \zeta^{s_0} w_{n+1,k}, \\ e_1 w_{n,k} &= -q^{2n} [k]_q w_{n,k-1} + \zeta^{s_1} [n]_q [\mu - n + 1]_q w_{n-1,k}. \end{aligned}$$

Thus, we have an increasing filtration

$$\{0\} = (W_{\zeta_1}^+ \otimes \overline{W}_{\zeta_2}^-)_{-1} \subset (W_{\zeta_1}^+ \otimes \overline{W}_{\zeta_2}^-)_0 \subset (W_{\zeta_1}^+ \otimes \overline{W}_{\zeta_2}^-)_1 \subset \dots$$

formed by the submodules

$$(W_{\zeta_1}^+ \otimes \overline{W}_{\zeta_2}^-)_k = \bigoplus_{\ell=0}^k \bigoplus_{n=0}^{\infty} \mathbb{C} w_{n,\ell}$$

with the quotient modules

$$(W_{\zeta_1}^+ \otimes \overline{W}_{\zeta_2}^-)_k / (W_{\zeta_1}^+ \otimes \overline{W}_{\zeta_2}^-)_{k-1} \cong \widetilde{V}_{\zeta}^{\mu}[\zeta_k]. \quad (5.28)$$

Here $\zeta_k \in \widetilde{\mathfrak{h}}^*$ are determined by the relations

$$\zeta_k(h_0) = \mu + 2k + 2, \quad \zeta_k(h_1) = -\mu - 2k - 2, \quad (5.29)$$

compare (5.23) and (5.24) with (4.69).

5.1.4. *Tensor product of representations $\overline{\rho}_{\zeta_2}^-$ and $\rho_{\zeta_1}^+$.* Finally, we consider the tensor product of the representations $\overline{\rho}_{\zeta_2}^-$ and $\rho_{\zeta_1}^+$. Here we see that

$$q^{\nu h_0} w = q^{-2\nu(N_A - N_B)} w, \quad q^{\nu h_1} w = q^{2\nu(N_A - N_B)} w,$$

and that

$$\begin{aligned} e_0 w &= (\kappa_q^{-1} \zeta_2^{s_0} b_A q^{-N_A} + \zeta_1^{s_0} b_B^{\dagger} q^{2N_A}) w, \\ e_1 w &= (\zeta_2^{s_1} b_A^{\dagger} + \kappa_q^{-1} \zeta_1^{s_1} b_B q^{-2N_A - N_B}) w \end{aligned}$$

for any $w \in \overline{W}^- \otimes W^+$. To construct a convenient basis, we introduce an operator f acting on a vector $w \in \overline{W}^- \otimes W^+$ as

$$f w = (\zeta_1^{s_1} b_A q^{-N_A - 2N_B} + \kappa_q \zeta_2^{s_1} b_B^{\dagger}) w.$$

One can verify that

$$(e_0 f - f e_0) w = 0, \quad (e_1 f - f e_1) w = 0.$$

The basis in question is formed by the vectors

$$w_{n,k} = ((\zeta_1 \zeta_2)^{-s_0/2} e_0)^n f^k (u_0 \otimes v_0),$$

and one can show that

$$q^{\nu h_0} w_{n,k} = q^{2\nu(n+k+1)} w_{n,k}, \quad (5.30)$$

$$q^{\nu h_1} w_{n,k} = q^{-2\nu(n+k+1)} w_{n,k}, \quad (5.31)$$

$$e_0 w_{n,k} = (\zeta_1 \zeta_2)^{s_0/2} w_{n+1,k}, \quad (5.32)$$

$$\begin{aligned} e_1 w_{n,k} &= \kappa_q^{-1} (\zeta_1 \zeta_2)^{-s_0/2} q^{-n} (\zeta_1^s - q^{2n} \zeta_2^s) [n]_q w_{n-1,k} \\ &\quad + \kappa_q^{-1} q^{-1} [n]_q [n-1]_q w_{n-2,k+1}. \end{aligned} \quad (5.33)$$

As before, the first three relations are evident, and to prove the last one we define the operators x and y by (5.19) and (5.9). These operators act on a vector $w \in \overline{W}^- \otimes W^+$ as

$$xw = (\kappa_q^{-1}(q + q^{-1}) \zeta_1^s q^{-2N_B} - \kappa_q^{-1} q (\zeta_1^s + \zeta_2^s) - \kappa_q (q + q^{-1}) q^2 \zeta_1^{s_0} \zeta_2^{s_1} b_A^\dagger b_B^\dagger q^{2N_A}) w$$

and

$$yw = ((q + q^{-1}) q \zeta_1^{s_0} \zeta_2^s b_B^\dagger q^{2N_A} - (q + q^{-1}) q^{-1} \zeta_1^{s+s_0} b_B^\dagger q^{2N_A-2N_B} + \kappa_q^2 (q + q^{-1}) q^3 \zeta_1^{2s_0} \zeta_2^{s_1} b_A^\dagger (b_B^\dagger)^2 q^{4N_A}) w.$$

One can verify that

$$[x, f] = 0, \quad [y, f] = 0,$$

and, having in mind equation (5.20), we see that we only need to determine the action of e_1 , x and y on the vector $w_{0,0}$. The explicit form of the action of these operators on an arbitrary vector $\overline{W}^- \otimes W^+$ implies that

$$e_1 w_{0,0} = 0,$$

$$x w_{0,0} = \kappa_q^{-1} q^{-1} (\zeta_1^s - q^2 \zeta_2^s) w_{0,0},$$

$$y w_{0,0} = -(q + q^{-1}) q^{-1} \zeta_1^s e_0 w_{0,0} + \kappa_q^{-1} (q + q^{-1}) q^{-1} (\zeta_1 \zeta_2)^{s_0} f w_{0,0}.$$

Now, using (5.20), one can be convinced in the validity of (5.33).

Introducing the parameters ζ and μ with the help of (5.27), we write equations (5.32) and (5.33) as

$$e_0 w_{n,k} = \zeta^{s_0} w_{n+1,k},$$

$$e_1 w_{n,k} = \zeta^{s_1} [n]_q [\mu - n + 1]_q w_{n-1,k} + \kappa_q^{-1} q^{-1} [n]_q [n - 1]_q w_{n-2,k+1}.$$

Thus, we have a decreasing filtration

$$\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+ = (\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+)_{-1} \supset (\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+)_{0} \supset (\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+)_{1} \supset \dots$$

with the submodules

$$(\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+)_{k} = \bigoplus_{\ell=k}^{\infty} \bigoplus_{n=0}^{\infty} \mathbb{C} w_{n,\ell}$$

and the quotient modules

$$(\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+)_{k} / (\overline{W}_{\zeta_2}^- \otimes W_{\zeta_1}^+)_{k+1} \cong \tilde{V}_{\zeta}^{\mu}[\xi_k], \quad (5.34)$$

where $\xi_k \in \tilde{\mathfrak{h}}^*$ are determined by relations (5.29).

5.2. Commutativity relations. First, it is worth to note that since for any $\nu \in \mathbb{C}$ the element $q^{\nu h_1}$ is an invertible group-like element of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ commuting with the twist element q^{th_1} , we have

$$[q^{\nu h_1}, \mathcal{T}_{\mu}(\zeta)] = 0, \quad [q^{\nu h_1}, \mathcal{Q}'(\zeta)] = 0, \quad [q^{\nu h_1}, \overline{\mathcal{Q}}'(\zeta)] = 0,$$

see sections 3.3.1 and 3.5.1.

As we noted before, there are functional relations which are due only to the fact that the universal transfer operators and universal Q -operators are constructed from the universal R -matrices. These are the commutativity relations for the universal transfer matrices

$$[\tilde{\mathcal{T}}_{\mu_1}(\zeta_1), \tilde{\mathcal{T}}_{\mu_2}(\zeta_2)] = 0, \quad [\mathcal{T}_{m_1}(\zeta_1), \mathcal{T}_{m_2}(\zeta_2)] = 0, \quad [\tilde{\mathcal{T}}_{\mu}(\zeta_1), \mathcal{T}_m(\zeta_2)] = 0,$$

see relation (3.22), and the commutativity of the universal transfer operators and the universal Q -operators

$$\begin{aligned} [\tilde{\mathcal{T}}_\mu(\zeta_1), \mathcal{Q}'(\zeta_2)] &= 0, & [\tilde{\mathcal{T}}_\mu(\zeta_1), \overline{\mathcal{Q}}'(\zeta_2)] &= 0, \\ [\mathcal{T}_m(\zeta_1), \mathcal{Q}'(\zeta_2)] &= 0, & [\mathcal{T}_m(\zeta_1), \overline{\mathcal{Q}}'(\zeta_2)] &= 0, \end{aligned}$$

see relation (3.29).

Another set of commutativity relations follows from the properties of the representations used to define the universal transfer operators and universal Q -operators. Note that relations (5.3)–(5.6) are symmetric with respect to ζ_1 and ζ_2 . It is not difficult to understand that this fact implies the equation

$$[\mathcal{Q}'(\zeta_1), \mathcal{Q}'(\zeta_2)] = 0.$$

Similarly, relations (5.15)–(5.14) are symmetric with respect to ζ_1 and ζ_2 , therefore,

$$[\overline{\mathcal{Q}}'(\zeta_1), \overline{\mathcal{Q}}'(\zeta_2)] = 0.$$

Further, comparing (5.28) and (5.34) we conclude that

$$[\mathcal{Q}'(\zeta_1), \overline{\mathcal{Q}}'(\zeta_2)] = 0.$$

5.3. Universal TQ -relations. It follows from relations (5.28) and (4.68) that

$$\begin{aligned} \mathcal{Q}'(q^{(\mu+1)/s}\zeta)\overline{\mathcal{Q}}'(q^{-(\mu+1)/s}\zeta) \\ = \tilde{\mathcal{T}}_\mu(\zeta) q^{-(\mu/2+1)(h_1+2\phi)} \sum_{k=0}^{\infty} q^{-(h_1+2\phi)k} = \tilde{\mathcal{T}}_\mu(\zeta) \frac{q^{-(\mu/2+1)(h_1+2\phi)}}{1 - q^{-h_1-2\phi}}, \end{aligned}$$

where the parameters ζ and μ are defined by relations (5.27). We can write

$$\tilde{\mathcal{T}}_\mu(\zeta) = \mathcal{C} q^{(\mu+1)(h_1/2+\phi)} \mathcal{Q}'(q^{(\mu+1)/s}\zeta) \overline{\mathcal{Q}}'(q^{-(\mu+1)/s}\zeta), \quad (5.35)$$

where

$$\mathcal{C} = q^{h_1/2+\phi} - q^{-h_1/2-\phi}.$$

It is convenient to redefine the universal Q -operators as

$$\mathcal{Q}(\zeta) = \zeta^{sh_1/4} \mathcal{Q}'(\zeta), \quad \overline{\mathcal{Q}}(\zeta) = \zeta^{-sh_1/4} \overline{\mathcal{Q}}'(\zeta).$$

In accordance with our convention (4.29), we assume that

$$\zeta^{vh_1} = q^{vuh_1}.$$

The new universal Q -operators commute with the universal transfer operators:

$$[\tilde{\mathcal{T}}_\mu(\zeta_1), \mathcal{Q}(\zeta_2)] = 0, \quad [\tilde{\mathcal{T}}_\mu(\zeta_1), \overline{\mathcal{Q}}(\zeta_2)] = 0, \quad (5.36)$$

$$[\mathcal{T}_m(\zeta_1), \mathcal{Q}(\zeta_2)] = 0, \quad [\mathcal{T}_m(\zeta_1), \overline{\mathcal{Q}}(\zeta_2)] = 0, \quad (5.37)$$

and among themselves:

$$[\mathcal{Q}(\zeta_1), \mathcal{Q}(\zeta_2)] = 0, \quad [\overline{\mathcal{Q}}(\zeta_1), \overline{\mathcal{Q}}(\zeta_2)] = 0, \quad (5.38)$$

$$[\mathcal{Q}(\zeta_1), \overline{\mathcal{Q}}(\zeta_2)] = 0. \quad (5.39)$$

Equation (5.35) takes the form

$$\tilde{\mathcal{T}}_\mu(\zeta) = q^{(\mu+1)\phi} \mathcal{C} \mathcal{Q}(q^{(\mu+1)/s}\zeta) \overline{\mathcal{Q}}(q^{-(\mu+1)/s}\zeta), \quad (5.40)$$

and we write

$$\tilde{\mathcal{T}}_\mu(q^v/s\zeta) = q^{(\mu+1)\phi} \mathcal{C} \mathcal{Q}(q^{(\mu+v+1)/s}\zeta) \overline{\mathcal{Q}}(q^{-(\mu-v+1)/s}\zeta). \quad (5.41)$$

Now, introducing new parameters

$$\alpha = \mu + 1 + \nu, \quad \beta = -(\mu + 1) + \nu, \quad (5.42)$$

so that

$$\mu = (\alpha - \beta)/2 - 1, \quad \nu = (\alpha + \beta)/2,$$

we come to the equation

$$\tilde{\mathcal{T}}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s}\zeta) = q^{(\alpha-\beta)\phi/2} \mathcal{C} \mathcal{Q}(q^{\alpha/s}\zeta) \overline{\mathcal{Q}}(q^{\beta/s}\zeta). \quad (5.43)$$

Using (5.40), we easily obtain that

$$q^{\gamma\phi/2} \tilde{\mathcal{T}}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s}\zeta) \mathcal{Q}(q^{\gamma/s}\zeta) = q^{\alpha\phi/2} \tilde{\mathcal{T}}_{(\gamma-\beta)/2-1}(q^{(\gamma+\beta)/2s}\zeta) \mathcal{Q}(q^{\alpha/s}\zeta) \quad (5.44)$$

and that

$$\begin{aligned} q^{-\gamma\phi/2} \tilde{\mathcal{T}}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s}\zeta) \overline{\mathcal{Q}}(q^{\gamma/s}\zeta) \\ = q^{-\beta\phi/2} \tilde{\mathcal{T}}_{(\alpha-\gamma)/2-1}(q^{(\alpha+\gamma)/2s}\zeta) \overline{\mathcal{Q}}(q^{\beta/s}\zeta). \end{aligned} \quad (5.45)$$

Introduce the universal transfer operators $\mathcal{T}_\mu(\zeta)$ defined with the help of the trace tr_μ given by equation (4.67). It is clear that

$$\mathcal{T}_\mu(\zeta) = \tilde{\mathcal{T}}_\mu(\zeta) - \tilde{\mathcal{T}}_{-\mu-2}(\zeta). \quad (5.46)$$

The universal transfer operators $\mathcal{T}_\mu(\zeta)$ possesses the evident property

$$\mathcal{T}_{-\mu-2}(\zeta) = -\mathcal{T}_\mu(\zeta).$$

This gives, in particular, that $\mathcal{T}_{-1} = 0$. It worth to note that, as follows from the explicit expression for the universal R -matrix, $\mathcal{T}_0 = 1$.

Equations (5.46) and (5.40) give

$$\begin{aligned} \mathcal{T}_\mu(\zeta) = \mathcal{C} \left[q^{(\mu+1)\phi} \mathcal{Q}(q^{(\mu+1)/s}\zeta) \overline{\mathcal{Q}}(q^{-(\mu+1)/s}\zeta) \right. \\ \left. - q^{-(\mu+1)\phi} \mathcal{Q}(q^{-(\mu+1)/s}\zeta) \overline{\mathcal{Q}}(q^{(\mu+1)/s}\zeta) \right]. \end{aligned} \quad (5.47)$$

In particular, for $\mu = 0$ we have the Wronskian-type relation

$$\mathcal{C} \left[q^\phi \mathcal{Q}(q^{1/s}\zeta) \overline{\mathcal{Q}}(q^{-1/s}\zeta) - q^{-\phi} \mathcal{Q}(q^{-1/s}\zeta) \overline{\mathcal{Q}}(q^{1/s}\zeta) \right] = 1.$$

It is easy to obtain from (5.47) the equation

$$\begin{aligned} \mathcal{T}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s}\zeta) = \mathcal{C} \left[q^{(\alpha-\beta)\phi/2} \mathcal{Q}(q^{\alpha/s}\zeta) \overline{\mathcal{Q}}(q^{\beta/s}\zeta) \right. \\ \left. - q^{(\beta-\alpha)\phi/2} \mathcal{Q}(q^{\beta/s}\zeta) \overline{\mathcal{Q}}(q^{\alpha/s}\zeta) \right], \end{aligned}$$

which implies that

$$\begin{aligned} q^{\gamma\phi/2} \mathcal{T}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s}\zeta) \mathcal{Q}(q^{\gamma/s}\zeta) + q^{\alpha\phi/2} \mathcal{T}_{(\beta-\gamma)/2-1}(q^{(\beta+\gamma)/2s}\zeta) \mathcal{Q}(q^{\alpha/s}\zeta) \\ + q^{\beta\phi/2} \mathcal{T}_{(\gamma-\alpha)/2-1}(q^{(\gamma+\alpha)/2s}\zeta) \mathcal{Q}(q^{\beta/s}\zeta) = 0 \end{aligned} \quad (5.48)$$

and

$$\begin{aligned} q^{-\gamma\phi/2} \mathcal{T}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s}\zeta) \overline{\mathcal{Q}}(q^{\gamma/s}\zeta) + q^{-\alpha\phi/2} \mathcal{T}_{(\beta-\gamma)/2-1}(q^{(\beta+\gamma)/2s}\zeta) \overline{\mathcal{Q}}(q^{\alpha/s}\zeta) \\ + q^{-\beta\phi/2} \mathcal{T}_{(\gamma-\alpha)/2-1}(q^{(\gamma+\alpha)/2s}\zeta) \overline{\mathcal{Q}}(q^{\beta/s}\zeta) = 0. \end{aligned} \quad (5.49)$$

We call equations (5.48) and (5.49) the *universal TQ-relations*. Putting

$$\alpha = \gamma - 2, \quad \beta = \gamma + 2,$$

we obtain the relations of more usual form,

$$\mathcal{T}(\zeta) \mathcal{Q}(\zeta) = q^\phi \mathcal{Q}(q^{2/s} \zeta) + q^{-\phi} \mathcal{Q}(q^{-2/s} \zeta), \quad (5.50)$$

$$\mathcal{T}(\zeta) \overline{\mathcal{Q}}(\zeta) = q^{-\phi} \overline{\mathcal{Q}}(q^{2/s} \zeta) + q^\phi \overline{\mathcal{Q}}(q^{-2/s} \zeta), \quad (5.51)$$

where

$$\mathcal{T}(\zeta) = \mathcal{T}_1(\zeta) = -\mathcal{T}_{-3}(\zeta).$$

5.4. Universal TT-relations. Using relation (5.43), we obtain from (5.44), or from (5.45), the equation

$$\begin{aligned} \tilde{\mathcal{T}}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s} \zeta) \tilde{\mathcal{T}}_{(\gamma-\delta)/2-1}(q^{(\gamma+\delta)/2s} \zeta) \\ = \tilde{\mathcal{T}}_{(\gamma-\beta)/2-1}(q^{(\gamma+\beta)/2s} \zeta) \tilde{\mathcal{T}}_{(\alpha-\delta)/2-1}(q^{(\alpha+\delta)/2s} \zeta). \end{aligned}$$

For the universal transfer operators $\mathcal{T}_\mu(\zeta)$ defined by (5.46) we obtain

$$\begin{aligned} \mathcal{T}_{(\alpha-\beta)/2-1}(q^{(\alpha+\beta)/2s} \zeta) \mathcal{T}_{(\gamma-\delta)/2-1}(q^{(\gamma+\delta)/2s} \zeta) \\ = \mathcal{T}_{(\alpha-\gamma)/2-1}(q^{(\alpha+\gamma)/2s} \zeta) \mathcal{T}_{(\beta-\delta)/2-1}(q^{(\beta+\delta)/2s} \zeta) \\ - \mathcal{T}_{(\beta-\gamma)/2-1}(q^{(\beta+\gamma)/2s} \zeta) \mathcal{T}_{(\alpha-\delta)/2-1}(q^{(\alpha+\delta)/2s} \zeta). \end{aligned}$$

We call these relations the *universal TT-relations*. There are two interesting special cases of these relations. In the first case we put

$$\alpha = \gamma + 2, \quad \beta = \delta + 2$$

and obtain

$$\mathcal{T}_\mu(q^{1/s} \zeta) \mathcal{T}_\mu(q^{-1/s} \zeta) = 1 + \mathcal{T}_{\mu-1}(\zeta) \mathcal{T}_{\mu+1}(\zeta), \quad (5.52)$$

where $\mu = (\gamma - \delta)/2 - 1$. In the second case we put

$$\alpha = \gamma + 2, \quad \beta = \gamma - 2$$

and obtain

$$\mathcal{T}(\zeta) \mathcal{T}_\mu(q^{-(\mu+1)/s} \zeta) = \mathcal{T}_{\mu+1}(q^{-\mu/s} \zeta) + \mathcal{T}_{\mu-1}(q^{-(\mu+2)/s} \zeta), \quad (5.53)$$

where again $\mu = (\gamma - \delta)/2 - 1$.

5.5. Six-vertex model. The six-vertex model arises when we use for the second factor of the tensor product $U_q(\mathcal{L}(\mathfrak{sl}_2)) \otimes U_q(\mathcal{L}(\mathfrak{sl}_2))$ the representation $\varphi_{\eta_1}^1 \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \varphi_{\eta_n}^1$. It is convenient to introduce in this case the transfer matrices

$$\mathbb{T}_\mu^{\text{P}}(\zeta | \eta_1, \dots, \eta_n) = q^{n/2} \prod_{i=1}^n \left[(\zeta \eta_i^{-1})^{-s/2} e^{-\lambda_2(q^\mu (\zeta \eta_i^{-1})^s) - \lambda_2(q^{-\mu-2} (\zeta \eta_i^{-1})^s)} \right] \mathbb{T}_\mu(\zeta | \eta_1, \dots, \eta_n),$$

being a Laurent polynomial in $\zeta^{s/2}$. Similarly, we define the Q-operators

$$\begin{aligned} \mathbb{Q}^{\text{P}}(\zeta | \eta_1, \dots, \eta_n) &= (1 - q^{-n-2\phi}) \\ &\times \prod_{i=1}^n \left[(\zeta \eta_i^{-1})^{-s/4} e^{-\lambda_2(q^{-1} (\zeta \eta_i^{-1})^s)} \right] \left(\prod_{i=1}^n \eta_i \right)^{-s\mathbb{S}/2n} \mathbb{Q}(\zeta | \eta_1, \dots, \eta_n) \end{aligned}$$

and

$$\begin{aligned} \overline{\mathbb{Q}}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) &= (1 - q^{-n+2\phi}) \\ &\quad \times \prod_{i=1}^n \left[(\zeta\eta_i^{-1})^{-s/4} e^{-\lambda_2(q^{-1}(\zeta\eta_i^{-1})^s)} \right] \left(\prod_{i=1}^n \eta_i \right)^{s\mathbb{S}/2n} \overline{\mathbb{Q}}(\zeta|\eta_1, \dots, \eta_n) \end{aligned}$$

also being Laurent polynomials in $\zeta^{s/2}$. Here we assume that

$$q^{v\mathbb{S}} = (\varphi_{\eta_1}^1 \otimes_{\Delta^{\text{op}}} \dots \otimes_{\Delta^{\text{op}}} \varphi_{\eta_n}^1)(q^{vh_1/2}),$$

and introduce the third prefactor to restore the invariance

$$\mathbb{Q}^{\mathbb{P}}(\zeta v|\eta_1 v, \dots, \eta_n v) = \mathbb{Q}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n), \quad \overline{\mathbb{Q}}^{\mathbb{P}}(\zeta v|\eta_1 v, \dots, \eta_n v) = \overline{\mathbb{Q}}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n).$$

It is clear that the introduced transfer matrices and Q -operators satisfy the commutativity relations which follow from (5.36)–(5.39). Now, starting from (5.50) and using identity (4.52), we obtain

$$\begin{aligned} \mathbb{T}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) \mathbb{Q}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) &= q^\phi \prod_{i=1}^n a((\zeta\eta_i^{-1})^{-s/2}) \mathbb{Q}^{\mathbb{P}}(q^{2/s}\zeta|\eta_1, \dots, \eta_n) \\ &\quad + q^{-\phi} \prod_{i=1}^n b((\zeta\eta_i^{-1})^{-s/2}) \mathbb{Q}^{\mathbb{P}}(q^{-2/s}\zeta|\eta_1, \dots, \eta_n), \end{aligned} \quad (5.54)$$

while relation (5.51) gives

$$\begin{aligned} \mathbb{T}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) \overline{\mathbb{Q}}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) &= q^{-\phi} \prod_{i=1}^n a((\zeta\eta_i^{-1})^{-s/2}) \overline{\mathbb{Q}}^{\mathbb{P}}(q^{2/s}\zeta|\eta_1, \dots, \eta_n) \\ &\quad + q^\phi \prod_{i=1}^n b((\zeta\eta_i^{-1})^{-s/2}) \overline{\mathbb{Q}}^{\mathbb{P}}(q^{-2/s}\zeta|\eta_1, \dots, \eta_n). \end{aligned}$$

Here the functions $a(\zeta)$ and $b(\zeta)$ are defined by (2.7). For $s = -2$ equation (5.54) coincides with the Baxter's TQ -equation (2.14).

Similarly, we obtain from (5.52) the relation

$$\begin{aligned} \mathbb{T}_\mu^{\mathbb{P}}(q^{1/s}\zeta|\eta_1, \dots, \eta_n) \mathbb{T}_\mu^{\mathbb{P}}(q^{-1/s}\zeta|\eta_1, \dots, \eta_n) &= \prod_{i=1}^n a(q^{\mu/2}(\zeta\eta_i^{-1})^{-s/2}) b(q^{-\mu/2}(\zeta\eta_i^{-1})^{-s/2}) \\ &\quad + \mathbb{T}_{\mu-1}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) \mathbb{T}_{\mu+1}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n), \end{aligned}$$

and from (5.53) the relation

$$\begin{aligned} \mathbb{T}^{\mathbb{P}}(\zeta|\eta_1, \dots, \eta_n) \mathbb{T}_\mu^{\mathbb{P}}(q^{-(\mu+1)/s}\zeta|\eta_1, \dots, \eta_n) &= \prod_{i=1}^n a((\zeta\eta_i^{-1})^{-s/2}) \mathbb{T}_{\mu+1}^{\mathbb{P}}(q^{-\mu/s}\zeta|\eta_1, \dots, \eta_n) \\ &\quad + \prod_{i=1}^n b((\zeta\eta_i^{-1})^{-s/2}) \mathbb{T}_{\mu-1}^{\mathbb{P}}(q^{-(\mu+2)/s}\zeta|\eta_1, \dots, \eta_n). \end{aligned}$$

Thus, we obtain all known functional relations for the six-vertex model as a consequence of the universal TQ - and TT -relations.

As we noted before, in the case of an even n the limit $\phi \rightarrow 0$ is singular for the matrices $\mathbb{Q}(\zeta|\eta_1, \dots, \eta_n)$ and $\overline{\mathbb{Q}}(\zeta|\eta_1, \dots, \eta_n)$. However, some linear combinations of these operators with matrix coefficients are finite. One can find such combinations using the observation made by Pronko [57] on the relation of transfer matrices and Q -operators, see the paper [13] for the limiting case $q = 1$.

6. CONCLUSIONS

We made an attempt to collect and organize general definitions and facts on the application of quantum groups to the construction of functional relations in the theory of integrable systems. As an example, we reconsidered the case of the quantum group $U_q(\mathcal{L}(\mathfrak{sl}_2))$ related to the six-vertex model. We proved the full set of the functional relations in the form independent of the representation of the quantum group in the quantum space and specialized them to the case of the six-vertex model. There are three sets of functional relations. The first set consists of commutativity relations satisfied by the universal transfer matrices and universal Q -operators. The second set is formed by the universal TQ -relations which are the origin of the Baxter's TQ -equations. The third set is formed by the universal TT -relations generating various fusion relations. The specialization of the universal TQ -relations and universal TT -relations to the case of the six-vertex model in the limiting case $q = 1$ was obtained by other methods in the papers [13, 33]. In fact, the universal TQ -relations and universal TT -relations have similar structures. It seems that they can be combined into one set of relations, see the paper [12] for the case of integrable systems related to Yangians $Y(\mathfrak{gl}_n)$.

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APPENDIX A. ENDOMORPHISM ALGEBRA

Let V be a finite-dimensional vector space of dimension n , and $\{e_a\}$ a basis of V . For any pair of indices a and b define an endomorphism $E_{ab} \in \text{End}(V)$ by the equation

$$E_{ab}e_c = e_a\delta_{bc}.$$

The endomorphisms E_{ab} satisfy the relation

$$E_{ab}E_{cd} = \delta_{bc}E_{ad}. \quad (\text{A.1})$$

One can verify that $\{E_{ab}\}$ is a basis of $\text{End}(V)$, so that any endomorphism $M \in \text{End}(V)$ has a unique representation of the form

$$M = \sum_{a,b} E_{ab}M_{ab}$$

for some $M_{ab} \in \mathbb{C}$. It is not difficult to see that

$$Me_a = \sum_b e_b M_{ba}.$$

One can consider M_{ab} as the entries of an $n \times n$ matrix called the matrix of M with respect to the basis $\{e_a\}$. From the other hand, any $n \times n$ matrix (M_{ab}) , via the above relation, defines an element of $\text{End}(V)$. This correspondence between the elements of $\text{End}(V)$ and $\text{Mat}_n(V)$ is a basis-dependent isomorphism of algebras.

In the case when $V = \mathbb{C}^n$ the algebra $\text{End}(V)$ is identified with the algebra $\text{Mat}_n(\mathbb{C})$. Here we assume that $\{e_a\}$ is the standard basis of \mathbb{C}^n . Hence, E_{ab} in this case are the standard matrix units.

Let V and U be finite-dimensional vector spaces, $\{e_a\}$ and $\{e_i\}$ their bases. One can show that $\{e_a \otimes e_i\}$ is a basis of the vector space $V \otimes U$. Let $\{E_{ai|bj}\}$ be a basis of $\text{End}(V \otimes U)$ corresponding to the basis $\{e_a \otimes e_i\}$, so that any endomorphism $M \in \text{End}(V \otimes U)$ can be uniquely represented as

$$M = \sum_{a,i,b,j} E_{ai|bj} M_{ai|bj}.$$

It is easy to see that

$$E_{ai|bj} = E_{ab} \otimes E_{ij}. \quad (\text{A.2})$$

This equation is a reflection of the natural isomorphism

$$\text{End}(V \otimes U) \cong \text{End}(V) \otimes \text{End}(U).$$

Now let V be a finite-dimensional vector space of dimension n , $\{e_a\}$ a basis of V , $\{E_{ab}\}$ the corresponding basis of $\text{End}(V)$, and A an algebra. It is clear that any element M of $\text{End}(V) \otimes A$ has a unique representation of the form

$$M = \sum_{a,b} E_{ab} \otimes M_{ab},$$

where M_{ab} are elements of A . As before, one can consider M_{ab} as the entries of $n \times n$ matrix called the matrix of the endomorphism M with respect to the basis $\{e_a\}$. Now it is an element of $\text{Mat}_n(A)$. Thus, we have a correspondence between the elements of $\text{End}(V) \otimes A$ and $\text{Mat}_n(A)$ which is an isomorphism of algebras. If A is the algebra $\text{End}(U)$ for some vector space U we obtain the isomorphism of the algebras $\text{End}(V \otimes U)$ and $\text{Mat}_n(\text{End}(U))$. Here one can uniquely represent a general element $w \in V \otimes U$ as

$$w = \sum_a e_a \otimes w_a,$$

where w_a are elements of U . One can consider w_a as the entries of $n \times 1$ matrix. Hence we can identify $V \otimes U$ with the module $\text{Mat}_{n,1}(U)$. Here the action of an element $M \in \text{End}(V \otimes U)$ on an element $w \in V \otimes U$ corresponds to matrix multiplication.

The similar consideration can be performed for the case $A \otimes \text{End}(V)$.

Introduce two useful operations for matrices with entries in algebras. First, let $M = (M_{ab}) \in \text{Mat}_n(A)$ and $N = (N_{ab}) \in \text{Mat}_n(B)$, where A and B are some algebras. We denote

$$M \boxdot N = \left(\sum_c M_{ac} \otimes N_{cb} \right). \quad (\text{A.3})$$

The matrix $M \boxdot N$ is an element of $\text{Mat}_n(A \otimes B)$.

Further, let A be an algebra, $K = (K_{ij}) \in \text{Mat}_m(A)$, and $L = (L_{rs}) \in \text{Mat}_\ell(A)$. Denote

$$K \boxtimes L = ((K \boxtimes L)_{ir|js}) = (K_{ij} L_{rs}). \quad (\text{A.4})$$

The operation \boxtimes is a natural generalization of the Kronecker product of matrices to the case of matrices with entries in a noncommutative algebra. It is clear that $K \boxtimes L \in \text{Mat}_{m\ell}(A)$.

APPENDIX B. SYMMETRIC GROUP AND TENSOR PRODUCTS

Let A_1, \dots, A_n be algebras, and

$$A = A_1 \otimes \dots \otimes A_n.$$

Given an element s of the symmetric group S_n , we define

$$A^s = A_{s^{-1}(1)} \otimes \dots \otimes A_{s^{-1}(n)}.$$

For any $t, s \in S_n$ we define Π^t as an isomorphism from A^s to A^{ts} by the equation

$$\Pi^t(a_1 \otimes \dots \otimes a_n) = a_{t^{-1}(1)} \otimes \dots \otimes a_{t^{-1}(n)}.$$

It is not difficult to show that

$$\Pi^{t_1} \circ \Pi^{t_2} = \Pi^{t_1 t_2}.$$

for all $t_1, t_2 \in S_n$.

Let j_1, j_2, \dots, j_m be distinct integers in the range from 1 to n , and M be an element of the tensor product $A_{j_1} \otimes \dots \otimes A_{j_m}$. Represent M as a sum

$$M = \sum_r a_1^r \otimes \dots \otimes a_m^r,$$

where $a_\ell^r \in A_{j_\ell}$. Let i_1, i_2, \dots, i_m be another set of distinct integers in the range from 1 to n . We denote by $M^{i_1 \dots i_m}$ the element of $A = A_1 \otimes \dots \otimes A_n$ which is the sum over r of monomials having for each $\ell = 1, \dots, m$ the element a_ℓ^r as the factor with the number i_ℓ , and 1 as all remaining factors. Here for any $s \in S_n$ we have

$$\Pi^s(M^{i_1 i_2 \dots i_m}) = M^{s(i_1) s(i_2) \dots s(i_m)}. \quad (\text{B.1})$$

We assume certainly that $A_{j_\ell} = A_{i_\ell}$.

Now let V_1, \dots, V_n be vector spaces, and

$$V = V_1 \otimes \dots \otimes V_n.$$

Given an element s of the symmetric group S_n , we define

$$V^s = V_{s^{-1}(1)} \otimes \dots \otimes V_{s^{-1}(n)}.$$

For any $t, s \in S_n$, we define an isomorphism P^t from V^s to V^{st} by

$$P^t(v_1 \otimes \dots \otimes v_n) = v_{t^{-1}(1)} \otimes \dots \otimes v_{t^{-1}(n)}.$$

In fact, the definitions of Π^t and P^t coincide. We use the notation Π^t when the tensor products of algebras are considered and P^t for the tensor products of vector spaces.

When $A_i = \text{End}(V_i)$, we have the relation

$$\Pi^t(M^{i_1 \dots i_k}) = P^t M^{i_1 \dots i_k} (P^t)^{-1} \quad (\text{B.2})$$

which implies the equation

$$P^t M^{i_1 \dots i_k} = M^{t(i_1) \dots t(i_k)} P^t. \quad (\text{B.3})$$

If t is a transposition (ij) we write Π^{ij} and P^{ij} instead of $\Pi^{(ij)}$ and $P^{(ij)}$ respectively. If $n = 2$ we denote $\Pi = \Pi^{12}$ and $P = P^{12}$.

Let $\varphi_1: A_1 \rightarrow B_1$ and $\varphi_2: A_2 \rightarrow B_2$ be homomorphisms of algebras. One can show that

$$\Pi \circ (\varphi_1 \otimes \varphi_2) = (\varphi_2 \otimes \varphi_1) \circ \Pi, \quad (\text{B.4})$$

where $\Pi \in \text{Hom}(B_1 \otimes B_2, B_2 \otimes B_1)$ at the left hand side of the equation and $\Pi \in \text{Hom}(A_1 \otimes A_2, A_2 \otimes A_1)$ at the right hand side.

APPENDIX C. QUASITRIANGULAR HOPF ALGEBRAS

Let A be a Hopf algebra with comultiplication Δ . One can show that A is also a Hopf algebra with comultiplication

$$\Delta^{\text{op}} = \Pi \circ \Delta. \quad (\text{C.1})$$

The Hopf algebra A is said to be *almost cocommutative* if there exists an invertible element $\mathcal{R} \in A \otimes A$ such that

$$\Delta^{\text{op}}(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1} \quad (\text{C.2})$$

for all $a \in A$. An almost cocommutative Hopf algebra A is called *quasitriangular* if

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{13} \mathcal{R}^{23}, \quad (\text{C.3})$$

$$(\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13} \mathcal{R}^{12}. \quad (\text{C.4})$$

In this case the element \mathcal{R} is called the *universal R-matrix*.

Write \mathcal{R} in the form $\mathcal{R} = \sum_i a_i \otimes b_i$. Multiplying both sides of equation (C.3) from the left by \mathcal{R}_{12} , we obtain

$$\begin{aligned} \mathcal{R}^{12} \mathcal{R}^{13} \mathcal{R}^{23} &= \mathcal{R}^{12} (\Delta \otimes \text{id})(\mathcal{R}) = \sum_i \mathcal{R} \Delta(a_i) \otimes b_i \\ &\stackrel{(\text{C.2})}{=} \sum_i \Delta^{\text{op}}(a_i) \mathcal{R} \otimes b_i = \left(\sum_i \Delta^{\text{op}}(a_i) \otimes b_i \right) (\mathcal{R} \otimes 1) = (\Delta^{\text{op}} \otimes \text{id})(\mathcal{R}) \mathcal{R}^{12}. \end{aligned}$$

Applying now the mapping Π^{12} to both sides of the same equation, we come to the relation

$$(\Delta^{\text{op}} \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{23} \mathcal{R}^{13}.$$

Hence, we see that the universal R -matrix satisfies the Yang-Baxter equation

$$\mathcal{R}^{12} \mathcal{R}^{13} \mathcal{R}^{23} = \mathcal{R}^{23} \mathcal{R}^{13} \mathcal{R}^{12}. \quad (\text{C.5})$$

 APPENDIX D. \mathbb{Z} -GRADED HOPF ALGEBRAS

If a Hopf algebra A is represented as a direct sum of linear subspaces

$$A = \bigoplus_{n \in \mathbb{Z}} A_n,$$

where

$$A_n A_m \subset A_{n+m}$$

and

$$\Delta(A_n) \subset \bigoplus_{m \in \mathbb{Z}} A_{n-m} \otimes A_m,$$

one says that A is \mathbb{Z} -graded. Note that A_0 is a subalgebra of A .

Any element a of a \mathbb{Z} -graded Hopf algebra A can be uniquely represented as a sum

$$a = \sum_{n \in \mathbb{Z}} a_n$$

with $a_n \in A_n$. Given $\nu \in \mathbb{C}^\times$, define a mapping $\Phi_\nu : A \rightarrow A$ by the equation

$$\Phi_\nu(a) = \sum_{n \in \mathbb{Z}} \nu^n a_n. \quad (\text{D.1})$$

It is clear that $a \in A_n$ if and only if $\Phi_\nu(a) = \nu^n a$. It is also not difficult to verify that

$$\Phi_{\nu_1 \nu_2} = \Phi_{\nu_1} \circ \Phi_{\nu_2}. \quad (\text{D.2})$$

Any \mathbb{Z} -gradation of a Hopf algebra A induces a \mathbb{Z} -gradation of the Hopf algebra $A \otimes A$, for which

$$(A \otimes A)_n = \bigoplus_{m \in \mathbb{Z}} A_{n-m} \otimes A_m = \bigoplus_{m \in \mathbb{Z}} A_m \otimes A_{n-m}.$$

Here the role of the automorphism Φ_ν is played by the automorphism $\Phi_\nu \otimes \Phi_\nu$.

APPENDIX E. TRACES ON ALGEBRAS

Recall that the usual trace of a matrix $M = (M_{ab}) \in \text{Mat}_n(\mathbb{C})$ is defined as

$$\text{tr}(M) = \sum_a M_{aa}.$$

The basic property of the trace is its cyclicity

$$\text{tr}(MN) = \text{tr}(NM)$$

for any two matrices $M, N \in \text{Mat}_n(\mathbb{C})$. Note also that for any two matrices $M \in \text{Mat}_n(\mathbb{C})$ and $N \in \text{Mat}_m(\mathbb{C})$ we have

$$\text{tr}(M \otimes N) = \text{tr}(M) \text{tr}(N). \quad (\text{E.1})$$

Let V be a finite-dimensional vector space, $\{e_a\}$ a basis of V , and $\{E_{ab}\}$ the corresponding basis of $\text{End}(V)$. The trace of an element $M = \sum_{a,b} E_{ab} M_{ab}$ of $\text{End}(V)$ can be defined as

$$\text{tr}(M) = \sum_a M_{aa}. \quad (\text{E.2})$$

It can be easily shown that $\text{tr}(M)$ does not depend on the choice of a basis. Hence, the trace of M coincides with the standard trace of its matrix with respect to any basis of V . We often denote this trace as tr_V . Here, the basic property of the trace holds, namely,

$$\text{tr}_V(MN) = \text{tr}_V(NM)$$

for all $M, N \in \text{End}(V)$. If U is another finite-dimensional vector space, then

$$\text{tr}_{V \otimes U}(M \otimes N) = \text{tr}_V(M) \text{tr}_U(N)$$

for all $M \in \text{End}(V)$ and $N \in \text{End}(U)$.

More generally, a trace on an algebra A is a linear mapping tr from A to \mathbb{C} , which satisfies the equation

$$\text{tr}(ab) = \text{tr}(ba) \quad (\text{E.3})$$

for all $a, b \in A$.

Multiplying a trace by a complex number we again obtain a trace. Up to this freedom the trace on $\text{End}(V)$, where V is a finite-dimensional vector space of dimension n given by (E.2), is unique. Indeed, assume that tr is a trace on $\text{End}(V)$. Relation (A.1) gives

$$E_{ac} E_{cb} = E_{ab},$$

therefore,

$$\text{tr}_V(E_{ac} E_{cb}) \stackrel{(\text{E.3})}{=} \text{tr}_V(E_{cb} E_{ac}) \stackrel{(\text{A.1})}{=} \delta_{ab} \text{tr}_V(E_{cc}) = \text{tr}_V(E_{ab}).$$

The last equation and the evident identity

$$\text{id}_V = \sum_c E_{cc}$$

give

$$n \text{tr}_V(E_{ab}) = \delta_{ab} \text{tr}_V(\text{id}_V),$$

and we obtain the trace defined by (E.2) if we assume that $\text{tr}(\text{id}_V) = n$.

Let tr_A and tr_B be traces on algebras A and B respectively. Then $\text{tr}_{A \otimes B} = \text{tr}_A \otimes \text{tr}_B$ is a trace on $A \otimes B$ and we have

$$\text{tr}_{A \otimes B}(a \otimes b) = \text{tr}_A(a)\text{tr}_B(b) \quad (\text{E.4})$$

for any $a \in A$ and $b \in B$. One can also consider the partial traces $\text{tr}_A \otimes \text{id}$ and $\text{id} \otimes \text{tr}_B$. The basic property of trace does not hold here, however,

$$(\text{tr}_A \otimes \text{id})((a \otimes 1)c) = (\text{tr}_A \otimes \text{id})(c(a \otimes 1)) \quad (\text{E.5})$$

for any $a \in A$ and $c \in A \otimes B$, and

$$(\text{id} \otimes \text{tr}_B)((1 \otimes b)c) = (\text{id} \otimes \text{tr}_B)(c(1 \otimes b))$$

for any $b \in B$ and $c \in A \otimes B$.

If φ is a homomorphism from an algebra A to an algebra B and tr_B is a trace on B , then

$$\text{tr}_A = \text{tr}_B \circ \varphi$$

is a trace on A . In particular, if φ is a representation of A in a finite-dimensional vector space V , then the mapping $\text{tr}_V \circ \varphi$ is a trace on A . Note that there are traces on algebras which cannot be obtained in this way.

In the case where V is an infinite-dimensional vector space, the trace is not defined for all elements of $\text{End}(V)$. In particular, it is not defined for the identity mapping.

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FACHBEREICH C – PHYSIK, BERGISCHE UNIVERSITÄT WUPPERTAL, 42097 WUPPERTAL, GERMANY
E-mail address: boos@physik.uni-wuppertal.de

FACHBEREICH C – PHYSIK, BERGISCHE UNIVERSITÄT WUPPERTAL, 42097 WUPPERTAL, GERMANY
E-mail address: goehmann@physik.uni-wuppertal.de

FACHBEREICH C – PHYSIK, BERGISCHE UNIVERSITÄT WUPPERTAL, 42097 WUPPERTAL, GERMANY
E-mail address: kluemper@uni-wuppertal.de

INSTITUTE FOR NUCLEAR RESEARCH OF THE RUSSIAN ACADEMY OF SCIENCES, 60TH OCTOBER
 AVE 7A, 117312 MOSCOW, RUSSIA

Current address: Fachbereich C – Physik, Bergische Universität Wuppertal, 42097 Wuppertal, Germany

E-mail address: knirov@physik.uni-wuppertal.de

INSTITUTE FOR HIGH ENERGY PHYSICS, 142281 PROTVINO, MOSCOW REGION, RUSSIA

Current address: Max-Planck-Institut für Mathematik, Vivatsgasse, 7, 53111, Bonn, Germany

E-mail address: Alexander.Razumov@ihep.ru